

# Regular partitions of dual polar spaces

Ilaria Cardinali and Bart De Bruyn

October 7, 2009

## Abstract

We describe several classes of regular partitions of dual polar spaces and determine their associated parameters and eigenvalues. We also give some general methods for computing the eigenvalues of regular partitions of dual polar spaces.

**Keywords:** dual polar space, regular partition

**MSC2000:** 51A50, 15A18, 05B25

## 1 Introduction

### 1.1 Regular partitions of point-line geometries

Let  $X$  be a nonempty finite set and let  $R \subseteq X \times X$  be a relation on  $X$ .

A partition  $\mathcal{P} = \{X_1, X_2, \dots, X_k\}$  of  $X$  is called *right- $R$ -regular* if there exist constants  $r_{ij}$  ( $1 \leq i, j \leq k$ ) such that for every  $x \in X_i$ , there are precisely  $r_{ij}$  elements  $y \in X_j$  such that  $(x, y) \in R$ . The  $(k \times k)$ -th matrix whose  $(i, j)$ -th entry (i.e. the entry in row  $i \in \{1, \dots, k\}$  and column  $j \in \{1, \dots, k\}$ ) is equal to  $r_{ij}$  is denoted by  $R_{\mathcal{P}}$ . Let  $E_{\mathcal{P}}^r$  denote the multiset whose elements are the complex eigenvalues of  $R_{\mathcal{P}}$ , the multiplicity of an element  $\lambda$  of  $E_{\mathcal{P}}^r$  being equal to the algebraic multiplicity of  $\lambda$  regarded as eigenvalue of  $R_{\mathcal{P}}$ .

A partition  $\mathcal{P} = \{X_1, X_2, \dots, X_k\}$  is called *left- $R$ -regular* if there exist constants  $l_{ij}$  ( $1 \leq i, j \leq k$ ) such that for every  $y \in X_i$ , there are precisely  $l_{ij}$  elements  $x \in X_j$  such that  $(x, y) \in R$ . The  $(k \times k)$ -th matrix whose  $(i, j)$ -th entry ( $1 \leq i, j \leq k$ ) is equal to  $l_{ij}$  is denoted by  $L_{\mathcal{P}}$ . Let  $E_{\mathcal{P}}^l$  denote the multiset whose elements are the complex eigenvalues of  $L_{\mathcal{P}}$ , the multiplicity of an element  $\lambda$  of  $E_{\mathcal{P}}^l$  being equal to the algebraic multiplicity of  $\lambda$  regarded as eigenvalue of  $L_{\mathcal{P}}$ .

Notice that a partition  $\mathcal{P}$  of  $X$  is left- $R$ -regular if and only if  $\mathcal{P}$  is right- $R^{-1}$ -regular, where  $R^{-1} := \{(y, x) \mid (x, y) \in R\}$ . So, for symmetric relations  $R$ , right- $R$ -regular and left- $R$ -regular partitions are the same. In this case, we will talk about  *$R$ -regular partitions* and we will denote  $R_{\mathcal{P}} = L_{\mathcal{P}}$  also by  $M_{\mathcal{P}}$ .

Given two finite multisets  $M = \{\lambda_1, \dots, \lambda_k\}$  and  $M' = \{\lambda'_1, \dots, \lambda'_{k'}\}$  whose elements are complex numbers, we denote by  $O(M, M')$  the multiplicity of 0 as an element of the multiset  $\{\lambda_i - \lambda'_j \mid 1 \leq i \leq k, 1 \leq j \leq k'\}$ .

For a proof of the following proposition, see De Wispelaere and Van Maldeghem [14, Lemma 3.3] (special case) or De Bruyn [9, Theorem 1.1].

**Proposition 1.1** *Let  $X$  be a nonempty finite set and let  $R \subseteq X \times X$  be a relation on  $X$ . Let  $\mathcal{P} = \{X_1, X_2, \dots, X_k\}$  be a partition of  $X$  which is right- $R$ -regular and let  $r_{ij}$  ( $1 \leq i, j \leq k$ ) denote the corresponding coefficients. Let  $\mathcal{P}' = \{X'_1, X'_2, \dots, X'_{k'}\}$  be a partition of  $X$  which is left- $R$ -regular and let  $l_{ij}$  ( $1 \leq i, j \leq k'$ ) denote the corresponding coefficients. Then the following holds:*

- (1)  $O(E_{\mathcal{P}}^r, E_{\mathcal{P}'}^l) \geq 1$ ;
- (2) If  $O(E_{\mathcal{P}}^r, E_{\mathcal{P}'}^l) = 1$ , then there exist numbers  $\eta_{ij}$ ,  $1 \leq i \leq k$  and  $1 \leq j \leq k'$ , only depending on the numbers  $r_{mn}$  ( $1 \leq m, n \leq k$ ) and  $l_{m'n'}$  ( $1 \leq m', n' \leq k'$ ) such that  $|X_i \cap X'_j| = \eta_{ij} \cdot |X|$ .
- (3) Suppose the following: (i)  $R$  is symmetric; (ii) there exists a  $\mu \in \mathbb{N} \setminus \{0\}$  such that for any  $x \in X$ , there are precisely  $\mu$  elements  $y \in X$  for which  $(x, y) \in R$ ; (iii)  $O(E_{\mathcal{P}}^r, E_{\mathcal{P}'}^l) = 1$ . Then  $|X_i \cap X'_j| = \frac{|X_i| \cdot |X'_j|}{|X|}$  for any  $(i, j) \in \{1, \dots, k\} \times \{1, \dots, k'\}$ .

Now, let  $\mathcal{S}$  be a point-line geometry with point-set  $X$  and symmetric collinearity relation  $R$ . So, if  $x_1, x_2 \in X$ , then  $(x_1, x_2) \in R$  if and only if  $x_1 \neq x_2$  and there is a line containing  $x_1$  and  $x_2$ . Put  $v = |X|$ ,  $X = \{p_1, p_2, \dots, p_v\}$  and let  $A$  be the  $(v \times v)$ -matrix whose  $(i, j)$ -th entry is equal to 1 if  $(p_i, p_j) \in R$  and equal to 0 otherwise. The eigenvalues of the collinearity matrix  $A$  are independent of the ordering of the points of  $X$  and are called the *eigenvalues of  $\mathcal{S}$* . The following lemma is known, see e.g. Godsfil [15, Section 5.2] or Godsfil and Royle [16, Theorem 9.3.3], but we add a proof since it is very short and easy.

**Lemma 1.2** *Suppose  $A$  can be partitioned as*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix},$$

where each  $A_{ii}$ ,  $i \in \{1, \dots, k\}$ , is square and each  $A_{ij}$ ,  $i, j \in \{1, \dots, k\}$ , has constant row sum  $b_{ij}$ . Then any eigenvalue of the matrix  $B = (b_{ij})_{1 \leq i, j \leq k}$  is also an eigenvalue of  $A$ .

**Proof.** Suppose  $A_{ij}$ ,  $i, j \in \{1, \dots, k\}$ , is a  $(v_i \times v_j)$ -matrix. Then  $v = v_1 + v_2 + \dots + v_k$ . Suppose  $Y$  is a  $(k \times 1)$ -matrix which is an eigenvector of  $B$  corresponding to some eigenvalue  $\lambda$  of  $B$ . Let  $Y'$  denote the  $(v \times 1)$ -matrix obtained from  $Y$  by replacing the  $(i, 1)$ -entry  $y_i$  of  $Y$  by the  $(v_i \times 1)$ -matrix with all entries equal to  $y_i$ . It is straightforward to verify that  $Y'$  is an eigenvector of  $A$  corresponding to the same eigenvalue  $\lambda$ . ■

Now, suppose that  $\mathcal{P} = \{X_1, \dots, X_k\}$  is an  $R$ -regular partition of the set  $X$ . We will also say that  $\mathcal{P}$  is a *regular* or *equitable partition* of  $\mathcal{S}$ . We may without loss of generality

suppose that we have ordered the points of  $X = \{p_1, \dots, p_v\}$  in such a way that if  $p_i \in X_l$  and  $p_j \in X_m$  with  $l < m$ , then  $i < j$ . If we partition the matrix  $A$  in submatrices  $A_{ij}$ ,  $i, j \in \{1, \dots, k\}$ , such that  $A_{ij}$  has dimension  $|X_i| \times |X_j|$ , then each of these submatrices has constant row sum. Lemma 1.2 then implies that every eigenvalue of  $B = M_{\mathcal{P}}$  is an eigenvalue of  $\mathcal{S}$ . We can say more:

**Lemma 1.3** *The matrix  $B$  is diagonalizable.*

**Proof.** Counting in two different ways the number of pairs  $(x, y) \in R$  where  $x \in X_i$  and  $y \in X_j$  yields  $|X_i| \cdot b_{ij} = |X_j| \cdot b_{ji}$  for all  $i, j \in \{1, \dots, k\}$ . Now, let  $Q$  be the diagonal matrix of size  $k \times k$  whose  $(i, i)$ -th entry is equal to  $\sqrt{|X_i|}$  ( $i \in \{1, \dots, k\}$ ). The fact that  $|X_i| \cdot b_{ij} = |X_j| \cdot b_{ji}$  for all  $i, j \in \{1, \dots, k\}$  implies that  $Q \cdot B \cdot Q^{-1}$  is a symmetric matrix. Hence,  $Q \cdot B \cdot Q^{-1}$  and  $B$  are diagonalizable. ■

Many regular partitions of a given point-line geometry  $\mathcal{S}$  are associated with nice substructures of  $\mathcal{S}$ , see [14] for the case of generalized hexagons. The eigenvalues of a regular partition might carry some combinatorial information on the substructures from which they are derived. Indeed, as Proposition 1.1(3) indicates, the knowledge of the eigenvalues is a helpful tool for calculating intersection sizes for all kind of combinatorial structures. In the present paper, we make a study of regular partitions of finite dual polar spaces.

## 1.2 Regular partitions of dual polar spaces

Dual polar spaces are important examples of point-line geometries which are related to polar spaces (see Cameron [3] and Tits [23, Chapter 7]).

Let  $\Pi$  be one of the following polar spaces of rank  $n \geq 2$ :

- (1) the symplectic polar space  $W(2n-1, q)$  whose singular subspaces are the subspaces of the projective space  $\text{PG}(2n-1, q)$  which are totally isotropic with respect to a given symplectic polarity of  $\text{PG}(2n-1, q)$ ;
- (2) the parabolic polar space  $Q(2n, q)$  whose singular subspaces are the subspaces of  $\text{PG}(2n, q)$  which are contained in a given nonsingular parabolic quadric of  $\text{PG}(2n, q)$ ;
- (3) the hyperbolic polar space  $Q^+(2n-1, q)$  whose singular subspaces are the subspaces of  $\text{PG}(2n-1, q)$  which are contained in a given nonsingular hyperbolic quadric of  $\text{PG}(2n-1, q)$ ;
- (4) the elliptic polar space  $Q^-(2n+1, q)$  whose singular subspaces are the subspaces of  $\text{PG}(2n+1, q)$  which are contained in a given nonsingular elliptic quadric of  $\text{PG}(2n+1, q)$ ;
- (5) the Hermitian polar space  $H(2n-1, q)$ ,  $q$  square, whose singular subspaces are the subspaces of  $\text{PG}(2n-1, q)$  which are contained in a given nonsingular Hermitian variety of  $\text{PG}(2n-1, q)$ ;
- (6) the Hermitian polar space  $H(2n, q)$ ,  $q$  square, whose singular subspaces are the subspaces of  $\text{PG}(2n, q)$  which are contained in a given nonsingular Hermitian variety of  $\text{PG}(2n, q)$ .

$\Pi$	$\Delta$	$e$	Quads
$W(2n-1, q)$	$DW(2n-1, q)$	1	$Q(4, q)$
$Q(2n, q)$	$DQ(2n, q)$	1	$W(q) = W(3, q)$
$Q^+(2n-1, q)$	$DQ^+(2n-1, q)$	0	$DQ^+(3, q)$
$Q^-(2n+1, q)$	$DQ^-(2n+1, q)$	2	$H(3, q^2)$
$H(2n-1, q)$	$DH(2n-1, q)$	$\frac{1}{2}$	$Q^-(5, \sqrt{q})$
$H(2n, q)$	$DH(2n, q)$	$\frac{3}{2}$	$DH(4, q)$

Table 1: The considered (dual) polar spaces

In each of the above cases,  $q$  is some prime power. Notice that the polar spaces  $W(2n-1, q)$  and  $Q(2n, q)$  are isomorphic if and only if  $q$  is even. With  $\Pi$  there is associated a *dual polar space*  $\Delta$  of rank  $n$ . This is the point-line geometry whose points, respectively lines, are the  $(n-1)$ -dimensional, respectively  $(n-2)$ -dimensional, singular subspaces of  $\Pi$ , with incidence being reverse containment. We will denote a dual polar space by putting a “D” in front of the name of the corresponding polar space. With each of the above (dual) polar spaces, we associate a parameter  $e$  as defined in Table 1.

The polar space  $\Pi$  has  $\frac{(q^n-1)(q^{e+n-1}+1)}{q-1}$  singular points and  $\prod_{i=0}^{n-1} (q^{e+i}+1)$  maximal singular subspaces. Every line of  $\Delta$  contains  $q^e+1$  points and every point of  $\Delta$  is contained in  $\frac{q^n-1}{q-1}$  lines. By Brouwer, Cohen and Neumaier [2, Theorem 9.4.3], the dual polar space  $\Delta$  has  $n+1$  eigenvalues, namely the numbers  $\lambda_j := q^e \frac{q^{n-j}-1}{q-1} - \frac{q^j-1}{q-1}$ ,  $j \in \{0, \dots, n\}$ . The multiplicity of the eigenvalue  $\lambda_j$ ,  $j \in \{0, \dots, n\}$ , is equal to  $f_j = q^j \begin{bmatrix} d \\ j \end{bmatrix}_q \frac{1+q^{d+e-2j}}{1+q^{d+e-j}} \prod_{i=1}^j \frac{1+q^{d+e-i}}{1+q^{i-e}}$ . Here,  $\begin{bmatrix} d \\ j \end{bmatrix}_q$  denotes the relevant Gaussian binomial coefficient.

The dual polar space  $\Delta$  is an example of a *near polygon* ([7]). This means that for every point  $x$  and every line  $L$ , there exists a unique point on  $L$  nearest to  $x$ . Here distances  $d(\cdot, \cdot)$  are measured in the collinearity graph of  $\Delta$ . Since the maximal distance between two points of  $\Delta$  is equal to  $n$ ,  $\Delta$  is a near  $2n$ -gon. For every point  $x$  of  $\Delta$  and every  $i \in \mathbb{N}$ ,  $\Delta_i(x)$  denotes the set of points of  $\Delta$  at distance  $i$  from  $x$ . For every point  $x$  of  $\Delta$ , we define  $x^\perp := \Delta_0(x) \cup \Delta_1(x)$ . If  $S$  is a subspace of  $\Delta$ , then we denote by  $\tilde{S}$  the subgeometry of  $\Delta$  induced on the point-set  $S$  by those lines of  $\Delta$  which are completely contained in  $S$ . If  $\alpha$  is a singular subspace of dimension  $n-1-k$  ( $k \in \{0, \dots, n\}$ ) of  $\Pi$ , then the set of all maximal singular subspaces of  $\Pi$  containing  $\alpha$  is a convex subspace  $F_\alpha$  of diameter  $k$  of  $\Delta$ . Conversely, every convex subspace of diameter  $k$  of  $\Delta$  is obtained in this way. In the sequel, we will say that  $\alpha$  is the singular subspace of  $\Pi$  corresponding to  $F_\alpha$ . If  $*_1, \dots, *_k$  are  $k \geq 1$  objects of  $\Delta$  (i.e. points or nonempty sets of points), then  $\langle *_1, \dots, *_k \rangle$  denotes the smallest convex subspace of  $\Delta$  containing  $*_1, \dots, *_k$ . The convex subspaces of  $\Delta$  through a given point  $x$  of  $\Delta$  define a projective space isomorphic to  $\text{PG}(n-1, q)$ . So, if  $F_i$ ,  $i \in \{1, 2\}$ , is a convex subspace of diameter  $n - \delta_i$  through  $x$ , then  $F_1 \cap F_2$  has diameter at least  $n - (\delta_1 + \delta_2)$ . Every two points  $x$  and  $y$  of  $\Delta$  at distance  $\delta \in \{0, \dots, n\}$  from each other are contained in a unique convex subspace of diameter  $\delta$ . These convex

subspaces are called *quads* if  $\delta = 2$ , *hexes* if  $\delta = 3$  and *maxes* if  $\delta = n - 1$ . In the last column of Table 1, we have mentioned the isomorphism class of the quads for each of the considered dual polar spaces. If  $F$  is a convex subspace of diameter  $\delta \in \{2, \dots, n\}$  of  $\Delta$ , then  $\tilde{F}$  is a dual polar space of rank  $\delta$  of the same type as  $\Delta$ . If  $x$  is a point and  $F$  is a convex subspace of  $\Delta$ , then  $F$  contains a unique point  $\pi_F(x)$  nearest to  $x$ . Moreover,  $d(x, y) = d(x, \pi_F(x)) + d(\pi_F(x), y)$  for every point  $y$  of  $F$ .

A regular partition of a finite dual polar space  $\Delta$  of rank  $n \geq 2$  can be constructed from any group  $G$  of automorphisms of  $\Delta$ . Indeed, the set of all  $G$ -orbits on the set of points of  $\Delta$  is an example of a regular partition. This suggests that many classes of regular partitions in a given finite dual polar space of rank  $n$  might exist. However, not all these regular partitions are probably interesting. For the application alluded to in Proposition 1.1(3) regular partitions having few eigenvalues (or at least, those having fewer than  $n + 1$  eigenvalues) are interesting.

The aim of this paper is to collect those classes of regular partitions of dual polar spaces which are related to nice geometrical substructures of dual polar spaces which have recently been under study in the literature. For each given regular partition, we also determine their parameters and eigenvalues. In Section 2 we give some general methods which will enable us to quickly compute the eigenvalues of many regular partitions.

The (dual) polar spaces of rank 2 are precisely the generalized quadrangles ([19]). Every regular partition of size 2 of a generalized quadrangle is either a so-called tight set or a so-called  $m$ -ovoid, see [1]. Tight sets and  $m$ -ovals of generalized quadrangles have already been studied in the literature, see the papers [1], [5], [6], [17] and [18]. Any regular partition of one of the generalized quadrangles occurring in the last column of Table 1 gives rise to regular partitions of dual polar spaces of rank  $n \geq 3$  by applying a construction which we will call extension (see Class 3 below).

In the following sections we will discuss several examples of regular partitions of dual polar spaces. We have divided these examples into 9 classes according to certain similarities in their constructions.

**Class 1.** This class contains some trivial examples of regular partitions. Additional regular partitions of dual polar spaces of rank  $n \geq 2$  can be derived from these examples by applying the construction mentioned in Class 3 (a few times).

(a) Let  $\Delta$  be one of the above-mentioned dual polar spaces of rank  $n \geq 2$  with parameters  $q$  and  $e$ , and let  $X$  denote the point set of  $\Delta$ . Then  $\{X\}$  is a regular partition of  $\Delta$  with eigenvalue  $\lambda_0 = q^e \frac{q^n - 1}{q - 1}$ .

(b) Let  $\Delta$  be one of the above-mentioned dual polar spaces of rank  $n \geq 2$  with parameters  $q$  and  $e$ . Let  $\mathcal{P}$  denote the partition of  $\Delta$  consisting of only singletons. Then  $\mathcal{P}$  is a regular partition of  $\Delta$  with associated eigenvalues  $\lambda_j = q^e \frac{q^{n-j} - 1}{q - 1} - \frac{q^j - 1}{q - 1}$ ,  $j \in \{0, \dots, n\}$ . The multiplicity of the eigenvalue  $\lambda_j$ ,  $j \in \{0, \dots, n\}$ , is equal to  $f_j = q^j \begin{bmatrix} d \\ j \end{bmatrix}_q \frac{1 + q^{d+e-2j}}{1 + q^{d+e-j}} \prod_{i=1}^j \frac{1 + q^{d+e-i}}{1 + q^{i-e}}$ .

(c) Let  $L$  be a line of size  $s + 1$  and  $\mathcal{P}$  a partition of  $L$  in  $k \geq 1$  subsets. Then  $\mathcal{P}$  is a regular partition with eigenvalues  $s$  (multiplicity 1) and  $-1$  (multiplicity  $k - 1$ ).

**Class 2.** This class contains examples of regular partitions which are related to sets of maxes of dual polar spaces. Let  $\Delta$  be one of the above-mentioned dual polar spaces of rank  $n \geq 2$  with parameters  $q$  and  $e$ .

(a) Let  $\{F_1^{(1)}, \dots, F_{n_1}^{(1)}, F_1^{(2)}, \dots, F_{n_2}^{(2)}, \dots, F_1^{(k)}, \dots, F_{n_k}^{(k)}\}$  be a set of mutually disjoint maxes of  $\Delta$ . Here,  $n_1, n_2, \dots, n_k$  are  $k \geq 1$  strictly positive integers. For every  $i \in \{1, \dots, k\}$ , put  $X_i := F_1^{(i)} \cup \dots \cup F_{n_i}^{(i)}$ . Let  $X_0$  denote the complement of  $X_1 \cup \dots \cup X_k$  in  $\Delta$ . If  $X_0 \neq \emptyset$ , then  $\{X_0, \dots, X_k\}$  is a regular partition of  $\Delta$  with associated eigenvalues  $\lambda_0 = q^{e \frac{q^n-1}{q-1}}$  (multiplicity 1) and  $\lambda_1 = q^{e \frac{q^{n-1}-1}{q-1}} - 1$  (multiplicity  $k$ ).

(b) Suppose  $X_1$  is a nonempty proper subspace of  $\Delta$  for which there exist nonzero constants  $\alpha$  and  $\beta$  such that through every point of  $X_1$  there are precisely  $\alpha$  lines and  $\beta$  maxes which are contained in  $X_1$ . Let  $X_2$  denote the complement of  $X_1$  in  $\Delta$ . Then  $\{X_1, X_2\}$  is a regular partition with eigenvalues  $\lambda_0 = q^{e \frac{q^n-1}{q-1}}$  and  $\lambda_1 = q^{e \frac{q^{n-1}-1}{q-1}} - 1$ .

(c) Suppose  $\Delta$  is not isomorphic to  $DQ^+(2n-1, q)$ . Let  $M_1$  and  $M_2$  denote two mutually disjoint maxes of  $\Delta$ . Put  $X_1 := M_1 \cup M_2$ . Let  $X_2$  denote the set of points of  $\Delta$  not contained in  $X_1$  which are contained on a line joining a point of  $M_1$  with a point of  $M_2$ . Let  $X_3$  be the set of points of  $\Delta$  not contained in  $X_1 \cup X_2$ . Then  $\{X_1, X_2, X_3\}$  is a regular partition of  $\Delta$  with eigenvalues  $\lambda_0 = q^{e \frac{q^n-1}{q-1}}$  (multiplicity 1) and  $\lambda_1 = q^{e \frac{q^{n-1}-1}{q-1}} - 1$  (multiplicity 2).

(d) Let  $M_1, M_2, X_2$  and  $X_3$  be as in (c). Then  $\{M_1, M_2, X_2, X_3\}$  is a regular partition of  $\Delta$  with eigenvalues  $\lambda_0 = q^{e \frac{q^n-1}{q-1}}$  (multiplicity 1) and  $\lambda_1 = q^{e \frac{q^{n-1}-1}{q-1}} - 1$  (multiplicity 3).

**Class 3.** We will now give a method for constructing new regular partitions from a given regular partition.

Let  $\Delta$  be one of the above-mentioned dual polar spaces of rank  $n \geq 2$  with parameters  $q$  and  $e$  and let  $F$  be a convex subspace of diameter  $n - \delta$ ,  $\delta \in \{0, \dots, n\}$ , of  $\Delta$ . Let  $\lambda_j = q^{e \frac{q^{n-\delta-j}-1}{q-1}} - \frac{q^j-1}{q-1}$ ,  $j \in \{0, \dots, n - \delta\}$ , denote the eigenvalues of  $\tilde{F}$  and let  $\lambda'_j = q^{e \frac{q^{n-j}-1}{q-1}} - \frac{q^j-1}{q-1}$ ,  $j \in \{0, \dots, n\}$ , denote the eigenvalues of  $\Delta$ . Let  $\mathcal{P} = \{X_1, \dots, X_k\}$  be a regular partition of  $\tilde{F}$ . For every  $i \in \{0, \dots, \delta\}$  and every  $j \in \{1, \dots, k\}$ , let  $Y_{ij}$  denote the set of points  $x$  at distance  $i$  from  $F$  for which  $\pi_F(x) \in X_j$ . Then  $\mathcal{P}' = \{Y_{ij} \mid 0 \leq i \leq \delta, 1 \leq j \leq k\}$  is a regular partition of  $\Delta$ . We call  $\mathcal{P}'$  the *extension of  $\mathcal{P}$* . If  $\lambda_{i_1}, \dots, \lambda_{i_k}$  are the  $k$  not necessarily distinct eigenvalues of the regular partition  $\mathcal{P}$ , then  $\lambda'_{i_l+j}$ ,  $l \in \{1, \dots, k\}$  and  $j \in \{0, \dots, \delta\}$ , are the  $k(\delta + 1)$  not necessarily distinct eigenvalues of the regular partition  $\mathcal{P}'$ .

We will now give a special case of the above. For every  $i \in \{0, \dots, \delta\}$ , let  $Z_i$  denote the set of points of  $\Delta$  at distance  $i$  from  $F$ . Then by example (1a) and the above,  $\{Z_0, Z_1, \dots, Z_\delta\}$  is a regular partition of  $\Delta$  with associated eigenvalues  $\lambda'_j = q^{e \frac{q^{n-j}-1}{q-1}} - \frac{q^j-1}{q-1}$ ,  $j \in \{0, \dots, \delta\}$ .

**Class 4.** The regular partitions of this class are related to sub-polar-spaces of polar spaces.

(a) Let  $\Pi$  be one of the above-mentioned polar spaces of rank  $n \geq 2$  with parameters  $q$  and  $e$ . Let  $\Pi'$  be a nondegenerate polar space of rank  $m \geq 2$  with parameters  $q$  and  $e'$

which is obtained from  $\Pi$  by intersecting it with a subspace  $\alpha$  of the ambient projective space of  $\Pi$ . Put  $k := \min(m, n + e - m - e')$ . Let  $X_i$ ,  $i \in \mathbb{N}$ , denote the set of maximal singular subspaces of  $\Pi$  which intersect  $\alpha$  in an  $(m - 1 - i)$ -dimensional singular subspace of  $\Pi'$ . Then  $\{X_0, X_1, \dots, X_k\}$  is a regular partition of the dual polar space  $\Delta$  associated with  $\Pi$ . The associated eigenvalues are  $\lambda_j = q^e \frac{q^{n-j}-1}{q-1} - \frac{q^j-1}{q-1}$ ,  $j \in \{0, \dots, k\}$ .

(b) Let  $\Pi$  be one of the following polar spaces of rank  $n \geq 2$ :  $Q(2n, q)$ ,  $Q^+(2n - 1, q)$ ,  $Q^-(2n + 1, q)$ . By Table 1, there is a parameter  $e \in \{0, 1, 2\}$  associated with  $\Pi$ . Let  $\alpha$  be a subspace of the ambient projective space of  $\Pi$  which intersects  $\Pi$  in a nonsingular hyperbolic quadric  $Q^+(2m - 1, q)$ ,  $m \geq 2$ . Let  $\mathcal{M}^+$  and  $\mathcal{M}^-$  denote the two families of maximal singular subspaces of  $Q^+(2m - 1, q)$ . Recall that two maximal singular subspaces of  $Q^+(2m - 1, q)$  belong to the same family if they intersect in a subspace of even co-dimension. Put  $k := \min(m, n + e - m)$ . Let  $X_i$ ,  $i \in \mathbb{N}$ , denote the set of maximal singular subspaces of  $\Pi$  which intersect  $\alpha$  in an  $(m - 1 - i)$ -dimensional singular subspace of  $Q^+(2m - 1, q)$ . Let  $X_0^\epsilon$ ,  $\epsilon \in \{+, -\}$ , denote the set of all maximal singular subspaces of  $\Pi$  which intersect  $\alpha$  in an element of  $\mathcal{M}^\epsilon$ . Then  $\{X_0^+, X_0^-, X_1, \dots, X_k\}$  is a regular partition of the dual polar space  $\Delta$  associated with  $\Pi$ . The associated eigenvalues are the numbers  $\lambda_j = q^e \frac{q^{n-j}-1}{q-1} - \frac{q^j-1}{q-1}$ ,  $j \in \{0, \dots, k\}$ , and the number  $\lambda_m = q^e \frac{q^{n-m}-1}{q-1} - \frac{q^m-1}{q-1}$ . So, if  $k = m$ , then the eigenvalue  $\lambda_m$  of the regular partition  $\{X_0^+, X_0^-, X_1, \dots, X_k\}$  has multiplicity 2.

**Class 5.** The regular partitions of this class are related to so-called SDPS-sets. These are sets of points of dual polar spaces satisfying certain nice properties, see Section 7.

Suppose  $n = 2m$  is even and let  $\Delta$  be one of the dual polar spaces  $DW(2n - 1, q)$ ,  $DQ^-(2n + 1, q)$ . Let  $X$  be an SDPS-set of  $\Delta$  and let  $X_i$ ,  $i \in \{0, \dots, m\}$ , denote the set of all points of  $\Delta$  at distance  $i$  from  $X$ . Then  $\{X_0, X_1, \dots, X_m\}$  is a regular partition of  $\Delta$  with associated eigenvalues  $\lambda_{2j} = q^e \frac{q^{n-2j}-1}{q-1} - \frac{q^{2j}-1}{q-1}$ ,  $j \in \{0, \dots, m\}$ . Recall here that  $e = 1$  if  $\Delta = DW(2n - 1, q)$  and  $e = 2$  if  $\Delta = DQ^-(2n + 1, q)$ .

**Class 6.** The regular partitions of this class are related to isometric embeddings of symplectic dual polar spaces into Hermitian dual polar spaces.

By De Bruyn [8], there exists up to isomorphism a unique set  $X$  of points of  $DH(2n - 1, q^2)$  satisfying: (i)  $\tilde{X} \cong DW(2n - 1, q)$ ; (ii)  $\tilde{X}$  is isometrically embedded into  $DH(2n - 1, q^2)$ , i.e. for all  $x_1, x_2 \in X$ , the distance between  $x_1$  and  $x_2$  in  $\tilde{X}$  coincides with the distance between  $x_1$  and  $x_2$  in  $DH(2n - 1, q^2)$ . Put  $m := \lfloor \frac{n}{2} \rfloor$ . Let  $X_i$ ,  $i \in \{0, \dots, m\}$ , denote the set of points of  $DH(2n - 1, q^2)$  at distance  $i$  from  $X$ . Then  $\{X_0, X_1, \dots, X_m\}$  is a regular partition of  $DH(2n - 1, q^2)$  with eigenvalues  $\lambda_j = q \frac{q^{2n-2j}-1}{q^2-1} - \frac{q^{2j}-1}{q^2-1}$ ,  $j \in \{0, \dots, m\}$ .

**Class 7.** The regular partitions of this class are related to hyperplanes of dual polar spaces. A *hyperplane* is a proper subspace which meets each line.

(a) By Shult [22], there exists a hyperplane  $X_1$  of  $DQ(6, q)$  such that  $\tilde{X}_1$  is isomorphic to the split-Cayley generalized hexagon  $H(q)$ . Such a hyperplane is called a *hexagonal hyperplane* of  $DQ(6, q)$ . Let  $X_2$  denote the complement of  $X_1$  in  $DQ(6, q)$ . Then  $\{X_1, X_2\}$  is a regular partition of  $DQ(6, q)$  with associated eigenvalues  $\lambda_0 = q^3 + q^2 + q$  and  $\lambda_2 = -1$ .

(b) Let  $Q^-(7, q)$  be a nonsingular elliptic quadric of  $\text{PG}(7, q)$ . Let  $\pi$  be a hyperplane of  $\text{PG}(7, q)$  which intersects  $Q^-(7, q)$  in a nonsingular parabolic quadric  $Q(6, q)$ . Let  $X_1$  be a set of singular planes of  $Q(6, q)$  which defines a hexagonal hyperplane of the dual polar space  $DQ(6, q)$  associated with  $Q(6, q)$ . Let  $X_2$  denote the set of singular planes of  $Q(6, q)$  not contained in  $X_1$ , let  $X_3$  denote the set of singular planes  $\alpha$  of  $Q^-(7, q)$  such that every singular plane of  $Q(6, q)$  through  $\alpha \cap \pi$  belongs to  $X_1$ , and let  $X_4$  denote the set of singular planes of  $Q^-(7, q)$  not belonging to  $X_1 \cup X_2 \cup X_3$ . By Pralle [20],  $X_1 \cup X_3$  is a hyperplane of  $DQ^-(7, q)$ . We will show that  $\{X_1, X_2, X_3, X_4\}$  is a regular partition of  $DQ^-(7, q)$  with eigenvalues  $\lambda_0 = q^4 + q^3 + q^2$ ,  $\lambda_1 = q^3 + q^2 - 1$ ,  $\lambda_2 = q^2 - q - 1$  and  $\lambda_3 = -q^2 - q - 1$ .

(c) Let  $Q(8, q)$  be a nonsingular parabolic quadric of  $\text{PG}(8, q)$  and let  $\pi$  be a hyperplane of  $\text{PG}(8, q)$  which intersects  $Q(8, q)$  in a nonsingular hyperbolic quadric  $Q^+(7, q)$  of  $\pi$ . Let  $X_1 := \mathcal{M}^+$  and  $\mathcal{M}^-$  be the two families of maximal singular subspaces of  $Q^+(7, q)$ . For every maximal singular subspace  $\alpha$  of  $Q(8, q)$  not contained in  $\pi$ , let  $\alpha^\epsilon$ ,  $\epsilon \in \{+, -\}$ , denote the unique element of  $\mathcal{M}^\epsilon$  through  $\alpha \cap \pi$ . Let  $HS^\epsilon(7, q)$  denote the half-spin geometry for  $Q^+(7, q)$  defined on the set  $\mathcal{M}^\epsilon$ . So, the points, respectively lines, of  $\mathcal{M}^\epsilon$  are the elements of  $\mathcal{M}^\epsilon$ , respectively the lines, of  $Q^+(7, q)$ , and incidence is reverse containment. The geometries  $HS^+(7, q)$  and  $HS^-(7, q)$  are isomorphic to the point-line system of the quadric  $Q^+(7, q)$ . So, there exists a hyperplane  $X_2$  of  $HS^-(7, q)$  whose points and lines define a polar space  $Q(6, q)$ . Let  $X_3$  denote the set of maximal singular subspaces of  $Q(8, q)$  not contained in  $\pi$  for which  $\alpha^- \in X_2$ . By Cardinali, De Bruyn and Pasini [4],  $X_1 \cup X_2 \cup X_3$  is a hyperplane of the dual polar space  $DQ(8, q)$  associated with  $Q(8, q)$ . Let  $X_4 \subseteq \mathcal{M}^-$  denote the set of maximal singular subspaces of  $Q^+(7, q)$  not belonging to  $X_1 \cup X_2$  and let  $X_5$  denote the set of maximal singular subspaces of  $Q(8, q)$  not belonging to  $X_1 \cup X_2 \cup X_3 \cup X_4$ . Then  $\{X_1, X_2, X_3, X_4, X_5\}$  is a regular partition of  $DQ(8, q)$  with eigenvalues  $\lambda_0 = q^4 + q^3 + q^2 + q$ ,  $\lambda_1 = q^3 + q^2 + q - 1$ ,  $\lambda_2 = q^2 - 1$ ,  $\lambda_3 = -q^2 - 1$  and  $\lambda_4 = -q^3 - q^2 - q - 1$ .

(d) Let  $H(5, q^2)$  denote a nonsingular Hermitian variety of  $\text{PG}(5, q^2)$  and let  $\zeta$  denote the Hermitian polarity of  $\text{PG}(5, q^2)$  associated with  $H(5, q^2)$ . Let  $\alpha$  be a plane of  $\text{PG}(5, q^2)$  satisfying the following properties: (i)  $\alpha$  and  $\alpha^\zeta$  are disjoint; (ii)  $\alpha \cap H(5, q^2)$  is a unital of  $\alpha$ ; (iii)  $\alpha^\zeta \cap H(5, q^2)$  is a unital of  $\alpha^\zeta$ . Let  $X_1$  denote the set of singular planes of  $H(5, q^2)$  which contain a point of  $\alpha$ , or equivalently, a point of  $\alpha^\zeta$ . By De Bruyn and Pralle [10], there exist precisely  $q + 1$  hyperplanes of  $DH(5, q^2)$  which arise from the so-called Grassmann-embedding of  $DH(5, q^2)$  which contain  $X_1$ . Let  $X_2$  be a set of points of  $DH(5, q^2)$  such that  $X_1 \cup X_2$  is one of these  $q + 1$  hyperplanes of  $DH(5, q^2)$ . Let  $X_3$  denote the complement of  $X_1 \cup X_2$  in  $DH(5, q^2)$ . Then  $\{X_1, X_2, X_3\}$  is a regular partition of  $DH(5, q^2)$  with eigenvalues  $\lambda_0 = q^5 + q^3 + q$ ,  $\lambda_1 = q^3 + q - 1$  and  $\lambda_2 = -q^2 + q - 1$ .

**Class 8.** The regular partitions of this class are related to isometric embeddings of some near hexagons in dual polar spaces of rank 3.

(a) Let  $X_1$  be subspace of  $DW(5, q)$  such that  $\widetilde{X}_1$  is a  $[(q + 1) \times (q + 1) \times (q + 1)]$ -cube which is isometrically embedded into  $DW(5, q)$ . Let  $X_2$ , respectively  $X_3$ , denote the set of points of  $DW(5, q)$  not contained in  $X_1$  which are collinear with precisely  $q + 1$ ,



respectively 1, point(s) of  $X_1$ . Then  $\{X_1, X_2, X_3\}$  is a regular partition of  $DW(5, q)$  with associated eigenvalues  $\lambda_0 = q^3 + q^2 + q$ ,  $\lambda_1 = q^2 + q - 1$  and  $\lambda_2 = -1$ .

(b) Let  $X_1$  be a subspace of  $DH(5, q^2)$  such that  $\widetilde{X_1}$  is a  $[(q+1) \times (q+1) \times (q+1)]$ -cube which is isometrically embedded into  $DH(5, q^2)$ . Let  $X_2, X_3$ , respectively  $X_4$ , denote the set of points of  $DH(5, q^2)$  not contained in  $X_1$  which are collinear with precisely  $q+1, 1$ , respectively 0, point(s) of  $X_1$ . Then  $\{X_1, X_2, X_3, X_4\}$  is a regular partition of  $DH(5, q^2)$  with associated eigenvalues  $\lambda_0 = q^5 + q^3 + q$  (multiplicity 1),  $\lambda_1 = q^3 + q - 1$  (multiplicity 2) and  $\lambda_2 = -q^2 + q - 1$  (multiplicity 1).

(c) The near hexagon  $Q(4, q) \times \mathbb{L}_{q+1}$  is obtained by taking  $q+1$  isomorphic copies of the generalized quadrangle  $Q(4, q)$  and joining the corresponding points to form lines of size  $q+1$ . Let  $X_1$  be a subspace of  $DH(5, q^2)$  such that  $\widetilde{X_1} \cong Q(4, q) \times \mathbb{L}_{q+1}$  is isometrically embedded into  $DH(5, q^2)$ . Let  $X_2, X_3$ , respectively  $X_4$ , denote the set of points of  $DH(5, q^2)$  not contained in  $X_1$  which are collinear with precisely  $q^2+1, q+1$ , respectively 1, point(s) of  $X_1$ . Then  $\{X_1, X_2, X_3, X_4\}$  is a regular partition of  $DH(5, q^2)$  with associated eigenvalues  $\lambda_0 = q^5 + q^3 + q$  (multiplicity 1),  $\lambda_1 = q^3 + q - 1$  (multiplicity 2) and  $\lambda_2 = -q^2 + q - 1$  (multiplicity 1).

**Class 9.** The regular partitions of this class are related to some “sporadic” isometric embeddings of near octagons in dual polar spaces of rank 4. We refer to [7] for the definition of the near octagons involved in the constructions.

(a) Let  $X_1$  be a subspace of  $DH(7, 4)$  such that  $\widetilde{X_1}$  is isomorphic to the near octagon  $\mathbb{H}_4$ . Suppose moreover that  $\widetilde{X_1}$  is isometrically embedded into  $DH(7, 4)$ . Let  $X_2, X_3, X_4, X_5$ , respectively  $X_6$ , denote the set of points of  $DH(7, 4)$  which are collinear with 7, 5, 3, 1, respectively 0, point(s) of  $X_1$ . Then  $\{X_1, X_2, X_3, X_4, X_5, X_6\}$  is a regular partition of  $DH(7, 4)$  with eigenvalues  $\lambda_0 = 170$ ,  $\lambda_1 = 41$  (multiplicity 2) and  $\lambda_2 = 5$  (multiplicity 3).

(b) Let  $X_1$  and  $X_2$  be as in (a). Then  $\widetilde{X_1 \cup X_2} \cong DW(7, 2)$  and  $\{X_1, X_2\}$  is a regular partition of  $\widetilde{X_1 \cup X_2} \cong DW(7, 2)$  with eigenvalues  $\lambda'_0 = 30$  and  $\lambda'_1 = 13$ .

(c) Let  $X_1$  be a subspace of  $DH(7, 4)$  such that  $\widetilde{X_1}$  is isomorphic to the near octagon  $\mathbb{G}_4$ . Suppose moreover that  $\widetilde{X_1}$  is isometrically embedded into  $DH(7, 4)$ . Let  $X_2$ , respectively  $X_3$ , denote the set of points of  $DH(7, 4)$  which are collinear with 15, respectively 7, points of  $X_1$ . Then  $\{X_1, X_2, X_3\}$  is a regular partition of  $DH(7, 4)$  with associated eigenvalues  $\lambda_0 = 170$ ,  $\lambda_1 = 41$  and  $\lambda_2 = 5$ .

**Merging.** If the parameters of a regular partition satisfy certain conditions, then a new regular partition can be derived from it by a procedure called *merging*.

Let  $\Delta$  be one of the above dual polar spaces of rank  $n \geq 2$ . Let  $\{X_1, X_2, \dots, X_k\}$  be a regular partition of  $\Delta$  with associated parameters  $a_{ij}$ ,  $1 \leq i, j \leq k$ . Let  $\{K_1, \dots, K_{k'}\}$  be a partition of the set  $\{1, \dots, k\}$ . For every  $i \in \{1, \dots, k'\}$ , put  $X'_i := \bigcup_{j \in K_i} X_j$ . If for all  $i_1, i_2 \in \{1, \dots, k'\}$ , the number  $b_{i_1, i_2} := \sum_{j_2 \in K_{i_2}} a_{j_1, j_2}$  is independent of the element  $j_1 \in K_{i_1}$ , then  $\{X'_1, X'_2, \dots, X'_{k'}\}$  is a regular partition of  $\Delta$  with associated parameters  $b_{i_1, i_2}$ ,  $1 \leq i_1, i_2 \leq k'$ . Several examples of merging can be given making use of the constructions given above. We will restrict ourselves to one example.

Consider a max  $M$  of  $\Delta$  and a point  $x \in M$ . Let  $X_i, i \in \{0, \dots, n-1\}$ , denote the set of points of  $M$  at distance  $i$  from  $x$ . Then  $\mathcal{P} = \{X_0, X_1, \dots, X_{n-1}\}$  is a regular partition of  $\widetilde{M}$  (see Class 3). Let  $Y_i, i \in \{0, \dots, n-1\}$ , denote the set of points of  $\Delta$  not contained in  $M$  for which  $\pi_M(x) \in X_i$ . Then the extension  $\mathcal{P}' = \{X_0, X_1, \dots, X_{n-1}, Y_0, Y_1, \dots, Y_{n-1}\}$  of  $\mathcal{P}$  is a regular partition of  $\Delta$ . Now, put  $X'_0 := X_0, X'_i := X_i \cup Y_{i-1}$  ( $i \in \{1, \dots, n-1\}$ ) and  $X'_n = Y_{n-1}$ . Then  $\{X'_0, X'_1, \dots, X'_n\}$  is a regular partition of  $\Delta$  since  $X'_i, i \in \{0, \dots, n\}$ , consists of all points of  $\Delta$  at distance  $i$  from  $x$ .

**Some applications.** Let  $\Delta$  be one of the above dual polar spaces of rank  $n \geq 2$ . As already said before, Proposition 1.1(3) can be used to determine the intersection size of two substructures  $X_1$  and  $X_2$  of  $\Delta$  if they belong to regular partitions  $\mathcal{P}_1$  and  $\mathcal{P}_2$  whose eigenvalues satisfy certain properties. We will now give a number of instances where Proposition 1.1(3) can be applied.

(1) Let  $\Delta$  be the dual polar space  $DQ(6, q)$ , let  $\mathcal{P}_1$  be a regular partition of  $\Delta$  associated with a hexagonal hyperplane of  $DQ(6, q)$  (see Class (7a)) and let  $\mathcal{P}_2$  be one of the regular partitions of  $\Delta$  belonging to Class 2.

(2) Let  $\Delta$  be the dual polar space  $DQ(6, q)$ , let  $\mathcal{P}_1$  be a regular partition of  $\Delta$  associated with a hexagonal hyperplane of  $DQ(6, q)$  and let  $\mathcal{P}_2$  be a regular partition of  $DQ(6, q)$  associated with one of the following sub-polar-spaces of  $\Pi = Q(6, q)$  (see Class (4a)):  $Q^+(5, q); Q(4, q)$ .

(3) Let  $\Delta$  be the dual polar space  $DQ(6, q)$ , let  $\mathcal{P}_1$  be a regular partition of  $\Delta$  associated with a hexagonal hyperplane of  $DQ(6, q)$  and let  $\mathcal{P}_2$  be a regular partition of  $DQ(6, q)$  associated with a sub-polar-space  $Q^+(5, q) \subseteq Q(6, q)$  as described in Class (4b).

(4) Let  $\Delta$  be one of the dual polar spaces  $DW(4n-1, q), DQ^-(4n+1, q), n \geq 1$ , and let  $X$  be an SDPS-set of  $\Delta$ . Let  $\mathcal{P}_1$  be the regular partition of  $\Delta$  associated with  $X$  (see Class 5) and let  $\mathcal{P}_2$  be one of the regular partitions of  $\Delta$  belonging to Class 2.

(5) Let  $\Delta$  be the dual polar space  $DW(4n-1, q), n \geq 2$ , and let  $X$  be an SDPS-set of  $\Delta$ . Let  $\mathcal{P}_1$  be the regular partition of  $\Delta$  associated with  $X$  and let  $\mathcal{P}_2$  denote a regular partition of  $\Delta$  associated with a sub-polar-space  $\Pi' \cong W(4n-3, q)$  of  $\Pi = W(4n-1, q)$  (see Class (4a)).

(6) Let  $\Delta$  be the dual polar space  $DW(4n-1, q) \cong DQ(4n, q), n \geq 1$  and  $q$  even, and let  $X$  be an SDPS-set of  $\Delta$ . Let  $\mathcal{P}_1$  be the regular partition of  $\Delta$  associated with  $X$ , and let  $\mathcal{P}_2$  denote a regular partition of  $\Delta$  which is associated with one of the following sub-polar-spaces of  $\Pi = Q(4n, q)$  (see Class (4a)):  $Q^+(4n-1, q); Q^-(4n-3, q) (n \geq 2); Q(4n-2, q) (n \geq 2)$ .

(7) Let  $\Delta$  be the dual polar space  $DQ^-(4n+1, q), n \geq 1$ , let  $X$  be an SDPS-set of  $\Delta$  and let  $\mathcal{P}_1$  be the regular partition of  $\Delta$  associated with  $X$ . Let  $\mathcal{P}_2$  be a regular partition of  $\Delta$  which is associated with one of the following sub-polar-spaces of  $\Pi = Q^-(4n+1, q)$  (see Class (4a)):  $Q^-(4n-1, q) (n \geq 2); Q(4n, q)$ .

(8) Let  $\Delta$  be the dual polar space  $DW(7, 2)$ , let  $X$  be an SDPS-set of  $\Delta$  and let  $\mathcal{P}_1$  be the regular partition of  $\Delta$  associated with  $X$ . Let  $\mathcal{P}_2$  be a regular partition of  $\Delta$  associated with a sub-near-octagon isomorphic to  $\mathbb{H}_4$  which is isometrically embedded into  $DW(7, 2)$  (see Class (9b)).

## 2 Methods for determining the eigenvalues of regular partitions of dual polar spaces

Let  $\mathcal{F}_n^{(i)}$ ,  $i \in \{1, 2, \dots, 6\}$ , be the family of dual polar spaces of rank  $n \geq 2$  whose associated polar spaces are as described in (i) of Section 1.2. For every prime power  $r$ , there corresponds a dual polar space  $\Delta_r \in \mathcal{F}_n^{(i)}$ . The value of  $q$  corresponding to  $\Delta_r$  is equal to  $r$  if  $i \in \{1, 2, 3, 4\}$  and equal to  $r^2$  if  $i \in \{5, 6\}$ .

The computation of the eigenvalues of a regular partition with a large number of classes can be a very difficult problem since it is equivalent to the computation of the eigenvalues of a square matrix with large dimension. In the case the regular partition belongs to a suitable infinite family of regular partitions, we will present solutions to this problem, see Corollaries 2.2, 2.4 and 2.6. In many cases, these corollaries will allow us to compute the eigenvalues with a minimal amount of effort. We will apply these results a number of times.

### 2.1 The case of quadratic and symplectic dual polar spaces

Suppose  $i \in \{1, 2, 3, 4\}$ . Then  $e \in \{0, 1, 2\}$ . For every  $j \in \{0, \dots, n\}$ , let  $\lambda_j(x)$  be the polynomial  $x^e \frac{x^{n-j}-1}{x-1} - \frac{x^j-1}{x-1} = x^e \cdot \sum_{i=0}^{n-j-1} x^i - \sum_{i=0}^{j-1} x^i$  of  $\mathbb{Z}[x]$ . Similarly, for every  $j \in \{0, \dots, n\}$ , let  $\lambda'_j(x)$  be the polynomial  $x^{e+n-j} - x^j$  of  $\mathbb{Z}[x]$ . Put  $N := \lfloor \frac{e+n}{2} \rfloor$ . Then we have

- $\{(\lambda'_j(x))^2 \mid 0 \leq j \leq n\} = \{(\lambda'_j(x))^2 \mid 0 \leq j \leq N\}$ ;
- $\deg(\lambda'_0(x)) > \deg(\lambda'_1(x)) > \dots > \deg(\lambda'_N(x))$ , where we have taken the convention that the zero polynomial of  $\mathbb{Z}[x]$  has degree  $-\infty$  (if  $e+n$  is even, then  $\lambda'_N(x) = 0$ ).

**Lemma 2.1** *Let  $k \geq 1$  and let  $a_{ij}(x) \in \mathbb{Z}[x]$  for all  $i, j \in \{1, \dots, k\}$ . Let  $\mathcal{R}$  be an infinite set of prime powers. Suppose that for every  $q \in \mathcal{R}$ , all eigenvalues of the matrix  $A(q) := (a_{ij}(q))_{1 \leq i, j \leq k}$  belong to the set  $\{\lambda_0(q), \lambda_1(q), \dots, \lambda_n(q)\}$ . Suppose also that  $i_1, \dots, i_k$  are  $k$  not necessarily distinct elements of  $\{0, \dots, n\}$  such that the following holds for every  $q \in \mathcal{R}$ :*

- (1)  $\lambda_{i_1}(q) + \lambda_{i_2}(q) + \dots + \lambda_{i_k}(q) = a_{11}(q) + a_{22}(q) + \dots + a_{kk}(q)$ ;
- (2)  $[\lambda_{i_1}(q)]^2 + [\lambda_{i_2}(q)]^2 + \dots + [\lambda_{i_k}(q)]^2 = [a_{11}(q)]^2 + [a_{22}(q)]^2 + \dots + [a_{kk}(q)]^2 + 2 \cdot \sum_{1 \leq l < l' \leq n} a_{ll'}(q) \cdot a_{l'l}(q)$ .

*Then for every  $q \in \mathcal{R}$ ,  $\lambda_{i_1}(q), \lambda_{i_2}(q), \dots, \lambda_{i_k}(q)$  are all the  $k$  not necessarily distinct eigenvalues of the matrix  $A(q)$ .*

**Proof.** We need to prove that the equation

$$\det(X \cdot I_k - A(q)) = (X - \lambda_{i_1}(q))(X - \lambda_{i_2}(q)) \cdots (X - \lambda_{i_k}(q)) \quad (1)$$

holds for every  $q \in \mathcal{R}$ . The right-hand and left-hand sides of equation (1) can be regarded as polynomials of degree  $k$  in  $X$  whose coefficients are polynomials in  $q$ . So, in order to prove (1) for every  $q \in \mathcal{R}$ , it suffices to prove that equation (1) holds for an infinite number of elements of  $\mathcal{R}$ . We will prove below that equation (1) holds for all prime powers  $q \in \mathcal{R}$  which are bigger than a certain number  $K$ .

Since  $\deg(\lambda'_0(x)) > \deg(\lambda'_1(x)) > \dots > \deg(\lambda'_N(x))$ , we know that there exists a  $K > 0$  such that  $\lambda'_i(q) > k \cdot \lambda'_{i+1}(q) \geq 0$  for every  $i \in \{0, \dots, N-1\}$  and every element  $q > K$  of  $\mathcal{R}$ . So, we also have  $(\lambda'_i(q))^2 > k \cdot (\lambda'_{i+1}(q))^2$  for every  $i \in \{0, \dots, N-1\}$  and every element  $q > K$  of  $\mathcal{R}$ .

Now, suppose  $q$  is some prime power of  $\mathcal{R}$  bigger than  $K$ . We know that there exist  $k$  not necessarily distinct elements  $j_1, \dots, j_k$  of  $\{0, \dots, n\}$  such that

$$\det(X \cdot I_k - A(q)) = (X - \lambda_{j_1}(q))(X - \lambda_{j_2}(q)) \cdots (X - \lambda_{j_k}(q)). \quad (2)$$

Equating the coefficients of  $X^{k-1}$  in the left-hand and right-hand sides of equation (2), we find that  $\lambda_{j_1}(q) + \lambda_{j_2}(q) + \dots + \lambda_{j_k}(q) = a_{11}(q) + a_{22}(q) + \dots + a_{kk}(q)$ . Hence,

$$\lambda_{i_1}(q) + \lambda_{i_2}(q) + \dots + \lambda_{i_k}(q) = \lambda_{j_1}(q) + \lambda_{j_2}(q) + \dots + \lambda_{j_k}(q). \quad (3)$$

Equating the coefficients of  $X^{k-2}$  in the left-hand and right-hand sides of equation (2), we find that  $\sum_{1 \leq l < l' \leq k} \lambda_{j_l}(q) \cdot \lambda_{j_{l'}}(q) = \sum_{1 \leq l < l' \leq k} (a_{ll}(q)a_{l'l'}(q) - a_{ll'}(q)a_{l'l}(q))$ , i.e.  $\frac{1}{2} \left( (\lambda_{j_1}(q) + \lambda_{j_2}(q) + \dots + \lambda_{j_k}(q))^2 - (\lambda_{j_1}(q))^2 - (\lambda_{j_2}(q))^2 - \dots - (\lambda_{j_k}(q))^2 \right) = \frac{1}{2} \left( (a_{11}(q) + a_{22}(q) + \dots + a_{kk}(q))^2 - (a_{11}(q))^2 - (a_{22}(q))^2 - \dots - (a_{kk}(q))^2 \right) - \sum_{1 \leq l < l' \leq k} a_{ll'}(q)a_{l'l}(q)$ . Together with  $\lambda_{j_1}(q) + \lambda_{j_2}(q) + \dots + \lambda_{j_k}(q) = a_{11}(q) + a_{22}(q) + \dots + a_{kk}(q)$ , this implies that  $[\lambda_{j_1}(q)]^2 + [\lambda_{j_2}(q)]^2 + \dots + [\lambda_{j_k}(q)]^2 = [a_{11}(q)]^2 + [a_{22}(q)]^2 + \dots + [a_{kk}(q)]^2 + 2 \cdot \sum_{1 \leq l < l' \leq k} a_{ll'}(q)a_{l'l}(q)$ . So, we have

$$[\lambda_{i_1}(q)]^2 + [\lambda_{i_2}(q)]^2 + \dots + [\lambda_{i_k}(q)]^2 = [\lambda_{j_1}(q)]^2 + [\lambda_{j_2}(q)]^2 + \dots + [\lambda_{j_k}(q)]^2. \quad (4)$$

Now, equations (3) and (4) imply that

$$\lambda'_{i_1}(q) + \lambda'_{i_2}(q) + \dots + \lambda'_{i_k}(q) = \lambda'_{j_1}(q) + \lambda'_{j_2}(q) + \dots + \lambda'_{j_k}(q) \quad (5)$$

and

$$[\lambda'_{i_1}(q)]^2 + [\lambda'_{i_2}(q)]^2 + \dots + [\lambda'_{i_k}(q)]^2 = [\lambda'_{j_1}(q)]^2 + [\lambda'_{j_2}(q)]^2 + \dots + [\lambda'_{j_k}(q)]^2. \quad (6)$$

Since  $(\lambda'_i(q))^2 > k \cdot (\lambda'_{i+1}(q))^2$  for every  $i \in \{0, \dots, N-1\}$ , there exist by equation (6) constants  $M_i$ ,  $i \in \{0, \dots, N\}$ , such that:

- (1) there are precisely  $M_i$  elements  $l \in \{1, \dots, k\}$  for which  $(\lambda'_{i_l}(q))^2 = (\lambda'_i(q))^2$ ;
- (2) there are precisely  $M_i$  elements  $l \in \{1, \dots, k\}$  for which  $(\lambda'_{j_l}(q))^2 = (\lambda'_i(q))^2$ .

Clearly,  $M_0 + \cdots + M_N = k$ . For every  $i \in \{0, \dots, N\}$ , let  $M'_i$ , respectively  $M''_i$ , denote the number of  $l \in \{1, \dots, k\}$  for which  $\lambda'_{i_l}(q) = \lambda'_i(q)$ , respectively  $\lambda'_{j_l}(q) = \lambda'_i(q)$ . Then equation (5) implies that  $(2M'_0 - M_0)\lambda'_0(q) + \cdots + (2M'_N - M_N)\lambda'_N(q) = (2M''_0 - M_0)\lambda'_0(q) + \cdots + (2M''_N - M_N)\lambda'_N(q)$ , i.e.  $M'_0\lambda'_0(q) + \cdots + M'_N\lambda'_N(q) = M''_0\lambda'_0(q) + \cdots + M''_N\lambda'_N(q)$ . Since  $M'_0 + \cdots + M'_N \leq k$ ,  $M''_0 + \cdots + M''_N \leq k$  and  $\lambda'_i(q) > k \cdot \lambda'_{i+1}(q) \geq 0$  for every  $i \in \{0, \dots, N-1\}$ , this implies that  $M'_0 = M''_0, M'_1 = M''_1, \dots, M'_N = M''_N$ . It follows that the multisets  $\{\lambda'_{i_1}(q), \lambda'_{i_2}(q), \dots, \lambda'_{i_k}(q)\}$  and  $\{\lambda'_{j_1}(q), \lambda'_{j_2}(q), \dots, \lambda'_{j_k}(q)\}$  are equal. Hence, also the multisets  $\{\lambda_{i_1}(q), \lambda_{i_2}(q), \dots, \lambda_{i_k}(q)\}$  and  $\{\lambda_{j_1}(q), \lambda_{j_2}(q), \dots, \lambda_{j_k}(q)\}$  are equal. This implies that

$$\begin{aligned} \det(X \cdot I_k - A(q)) &= (X - \lambda_{j_1}(q))(X - \lambda_{j_2}(q)) \cdots (X - \lambda_{j_k}(q)) \\ &= (X - \lambda_{i_1}(q))(X - \lambda_{i_2}(q)) \cdots (X - \lambda_{i_k}(q)). \end{aligned}$$

This is precisely what we needed to prove.  $\blacksquare$

The following is a corollary of Lemmas 1.2 and 2.1.

**Corollary 2.2** *Let  $k \geq 1$  and let  $a_{ij}(x) \in \mathbb{Z}[x]$  for all  $i, j \in \{1, \dots, k\}$ . Let  $\mathcal{R}$  be an infinite set of prime powers. Suppose that for every  $q \in \mathcal{R}$ ,  $\mathcal{P}_q = \{X_1(q), X_2(q), \dots, X_k(q)\}$  is a regular partition of  $\Delta_q$  with associated coefficients  $a_{ij}(q)$ ,  $1 \leq i, j \leq k$ . Suppose also that  $i_1, \dots, i_k$  are  $k$  not necessarily distinct elements of  $\{0, \dots, n\}$  such that the following holds for every  $q \in \mathcal{R}$ :*

- (1)  $\lambda_{i_1}(q) + \lambda_{i_2}(q) + \cdots + \lambda_{i_k}(q) = a_{11}(q) + a_{22}(q) + \cdots + a_{kk}(q);$
- (2)  $[\lambda_{i_1}(q)]^2 + [\lambda_{i_2}(q)]^2 + \cdots + [\lambda_{i_k}(q)]^2 = [a_{11}(q)]^2 + [a_{22}(q)]^2 + \cdots + [a_{kk}(q)]^2 + 2 \cdot \sum_{1 \leq l < l' \leq n} a_{ll'}(q) \cdot a_{l'l}(q).$

*Then for every  $q \in \mathcal{R}$ ,  $\lambda_{i_1}(q), \lambda_{i_2}(q), \dots, \lambda_{i_k}(q)$  are all the  $k$  not necessarily distinct eigenvalues of the regular partition  $\mathcal{P}_q$  of  $\Delta_q$ .*

**Lemma 2.3** *Let  $k \geq 1$  and let  $a_{ij}(x) \in \mathbb{Z}[x]$  for all  $i, j \in \{1, \dots, k\}$ . Let  $\mathcal{R}$  be an infinite set of prime powers. Suppose that for every  $q \in \mathcal{R}$ , all eigenvalues of the matrix  $A(q) := (a_{ij}(q))_{1 \leq i, j \leq k}$  belong to the set  $\{\lambda_0(q), \lambda_1(q), \dots, \lambda_n(q)\}$ . Suppose also that  $i_1, \dots, i_k$  are  $k$  not necessarily distinct elements of  $\{0, \dots, n\}$  satisfying:*

- (1) *There is at most one  $l \in \{1, \dots, k\}$  for which  $i_l = \frac{n+e}{2}$ .*
- (2) *There exists no two distinct elements  $l_1, l_2 \in \{1, \dots, k\}$  for which  $i_{l_1} + i_{l_2} = e + n$ .*
- (3)  $\lambda_{i_1}(q) + \lambda_{i_2}(q) + \cdots + \lambda_{i_k}(q) = a_{11}(q) + a_{22}(q) + \cdots + a_{kk}(q).$

*Then for every  $q \in \mathcal{R}$ ,  $\lambda_{i_1}(q), \lambda_{i_2}(q), \dots, \lambda_{i_k}(q)$  are all the  $k$  not necessarily distinct eigenvalues of the matrix  $A(q)$ .*

**Proof.** We need to prove that the equation

$$\det(X \cdot I_k - A(q)) = (X - \lambda_{i_1}(q))(X - \lambda_{i_2}(q)) \cdots (X - \lambda_{i_k}(q)) \quad (7)$$

holds for every  $q \in \mathcal{R}$ . The right-hand and left-hand sides of equation (7) can be regarded as polynomials of degree  $k$  in  $X$  whose coefficients are polynomials in  $q$ . So, in order to prove (7) for every  $q \in \mathcal{R}$ , it suffices to prove that equation (7) holds for an infinite number of elements of  $\mathcal{R}$ . We will prove below that equation (7) holds for all prime powers  $q \in \mathcal{R}$  which are bigger than a certain number  $K$ .

By conditions (1) and (2), the sum of the absolute values of the coefficients of the polynomial  $\lambda'_{i_1}(x) + \lambda'_{i_2}(x) + \cdots + \lambda'_{i_k}(x)$  is equal to either  $2k - 2$  or  $2k$  depending on whether there exists an  $l \in \{1, \dots, k\}$  for which  $i_l = \frac{e+n}{2}$  or not. This implies that if  $j_1, j_2, \dots, j_k$  are  $k$  not necessarily distinct elements of  $\{0, \dots, n\}$  such that the polynomials  $\lambda'_{i_1}(x) + \lambda'_{i_2}(x) + \cdots + \lambda'_{i_k}(x)$  and  $\lambda'_{j_1}(x) + \lambda'_{j_2}(x) + \cdots + \lambda'_{j_k}(x)$  of  $\mathbb{Z}[x]$  are equal, then the multisets  $\{i_1, i_2, \dots, i_k\}$  and  $\{j_1, j_2, \dots, j_k\}$  are also equal. This implies that there exists a  $K > 0$  such that if  $q > K$  is an element of  $\mathcal{R}$  and  $j_1, j_2, \dots, j_k$  are  $k$  elements of  $\{0, \dots, n\}$  such that the multisets  $\{i_1, i_2, \dots, i_k\}$  and  $\{j_1, j_2, \dots, j_k\}$  are distinct, then also  $\lambda'_{i_1}(q) + \lambda'_{i_2}(q) + \cdots + \lambda'_{i_k}(q)$  and  $\lambda'_{j_1}(q) + \lambda'_{j_2}(q) + \cdots + \lambda'_{j_k}(q)$  are distinct.

Now, suppose that  $q$  is some prime power of  $\mathcal{R}$  bigger than  $K$ . We know that there exist  $k$  not necessarily distinct elements  $j_1, \dots, j_k$  of  $\{0, \dots, n\}$  such that

$$\det(X \cdot I_k - A(q)) = (X - \lambda_{j_1}(q))(X - \lambda_{j_2}(q)) \cdots (X - \lambda_{j_k}(q)). \quad (8)$$

Equating the coefficients of  $X^{k-1}$  in the left-hand and right-hand sides of equation (8), we find that  $\lambda_{j_1}(q) + \lambda_{j_2}(q) + \cdots + \lambda_{j_k}(q) = a_{11}(q) + a_{22}(q) + \cdots + a_{kk}(q)$ . Hence,  $\lambda_{i_1}(q) + \lambda_{i_2}(q) + \cdots + \lambda_{i_k}(q) = \lambda_{j_1}(q) + \lambda_{j_2}(q) + \cdots + \lambda_{j_k}(q)$  and  $\lambda'_{i_1}(q) + \lambda'_{i_2}(q) + \cdots + \lambda'_{i_k}(q) = \lambda'_{j_1}(q) + \lambda'_{j_2}(q) + \cdots + \lambda'_{j_k}(q)$ . As said before the fact that  $q > K$  implies that the multisets  $\{i_1, i_2, \dots, i_k\}$  and  $\{j_1, j_2, \dots, j_k\}$  are equal. Equation (8) then implies that

$$\det(X \cdot I_k - A(q)) = (X - \lambda_{i_1}(q))(X - \lambda_{i_2}(q)) \cdots (X - \lambda_{i_k}(q)).$$

This is precisely what we needed to prove. ■

The following is a corollary of Lemmas 1.2 and 2.3.

**Corollary 2.4** *Let  $k \geq 1$  and let  $a_{ij}(x) \in \mathbb{Z}[x]$  for all  $i, j \in \{1, \dots, k\}$ . Let  $\mathcal{R}$  be an infinite set of prime powers. Suppose that for every  $q \in \mathcal{R}$ ,  $\mathcal{P}_q = \{X_1(q), X_2(q), \dots, X_k(q)\}$  is a regular partition of  $\Delta_q$  with associated coefficients  $a_{ij}(q)$ ,  $1 \leq i, j \leq k$ . Suppose also that  $i_1, \dots, i_k$  are  $k$  not necessarily distinct elements of  $\{0, \dots, n\}$  satisfying*

- (1) *There is at most one  $l \in \{1, \dots, k\}$  for which  $i_l = \frac{n+e}{2}$ .*
- (2) *There exists no two distinct elements  $l_1, l_2 \in \{1, \dots, k\}$  for which  $i_{l_1} + i_{l_2} = e + n$ .*
- (3)  $\lambda_{i_1}(q) + \lambda_{i_2}(q) + \cdots + \lambda_{i_k}(q) = a_{11}(q) + a_{22}(q) + \cdots + a_{kk}(q)$ .

*Then for every  $q \in \mathcal{R}$ ,  $\lambda_{i_1}(q), \lambda_{i_2}(q), \dots, \lambda_{i_k}(q)$  are all the  $k$  not necessarily distinct eigenvalues of the regular partition  $\mathcal{P}_q$  of  $\Delta_q$ .*

## 2.2 The case of Hermitian dual polar spaces

Suppose  $i \in \{5, 6\}$ . Then  $e \in \{\frac{1}{2}, \frac{3}{2}\}$ . For every  $j \in \{0, \dots, n\}$ , let  $\lambda_j(x)$  be the polynomial  $x^{2e} \frac{x^{2n-2j}-1}{x^2-1} - \frac{x^{2j}-1}{x^2-1} = x^{2e} \sum_{i=0}^{n-j-1} x^{2i} - \sum_{i=0}^{j-1} x^{2i}$  of  $\mathbb{Z}[x]$ . The degrees of the  $n+1$  nonzero polynomials  $\lambda_j(x)$ ,  $j \in \{0, \dots, n\}$ , are mutually distinct.

**Lemma 2.5** *Let  $k \geq 1$  and let  $a_{ij}(x) \in \mathbb{Z}[x]$  for all  $i, j \in \{1, \dots, k\}$ . Let  $\mathcal{R}$  be an infinite set of prime powers. Suppose that for every  $r \in \mathcal{R}$ , all eigenvalues of the matrix  $A(r) := (a_{ij}(r))_{1 \leq i, j \leq k}$  belong to the set  $\{\lambda_0(r), \lambda_1(r), \dots, \lambda_n(r)\}$ . Suppose also that  $i_1, \dots, i_k$  are  $k$  not necessarily distinct elements of  $\{0, \dots, n\}$  such that  $\lambda_{i_1}(r) + \lambda_{i_2}(r) + \dots + \lambda_{i_k}(r) = a_{11}(r) + a_{22}(r) + \dots + a_{kk}(r)$  for every  $r \in \mathcal{R}$ . Then for every  $r \in \mathcal{R}$ ,  $\lambda_{i_1}(r), \lambda_{i_2}(r), \dots, \lambda_{i_k}(r)$  are all the  $k$  not necessarily distinct eigenvalues of the matrix  $A(r)$ .*

**Proof.** We need to prove that the equation

$$\det(X \cdot I_k - A(r)) = (X - \lambda_{i_1}(r))(X - \lambda_{i_2}(r)) \cdots (X - \lambda_{i_k}(r)) \quad (9)$$

holds for every  $r \in \mathcal{R}$ . The right-hand and left-hand sides of equation (9) can be regarded as polynomials of degree  $k$  in  $X$  whose coefficients are polynomials in  $r$ . So, in order to prove (9) for every  $r \in \mathcal{R}$ , it suffices to prove that equation (9) holds for an infinite number of elements of  $\mathcal{R}$ . We will prove below that equation (9) holds for all prime powers  $r \in \mathcal{R}$  which are bigger than a certain number  $K$ .

Let  $\sigma$  be the permutation of  $\{0, \dots, n\}$  such that  $\deg(\lambda_{\sigma(0)}(x)) > \deg(\lambda_{\sigma(1)}(x)) > \dots > \deg(\lambda_{\sigma(n)}(x))$ . Obviously, there must exist a  $K > 0$  such that  $|\lambda_{\sigma(i)}(r)| > 2k \cdot |\lambda_{\sigma(i+1)}(r)|$  for every  $i \in \{0, \dots, n-1\}$  and every  $r > K$  of  $\mathcal{R}$ .

Now, suppose  $r$  is some prime power of  $\mathcal{R}$  bigger than  $K$ . We know that there exist  $k$  not necessarily distinct elements  $j_1, \dots, j_k$  of  $\{0, \dots, n\}$  such that

$$\det(X \cdot I_k - A(r)) = (X - \lambda_{j_1}(r))(X - \lambda_{j_2}(r)) \cdots (X - \lambda_{j_k}(r)). \quad (10)$$

Equating the coefficients of  $X^{k-1}$  in the left-hand and right-hand sides of equation (10), we find that  $\lambda_{j_1}(r) + \lambda_{j_2}(r) + \dots + \lambda_{j_k}(r) = a_{11}(r) + a_{22}(r) + \dots + a_{kk}(r)$ . Hence,

$$\lambda_{i_1}(r) + \lambda_{i_2}(r) + \dots + \lambda_{i_k}(r) = \lambda_{j_1}(r) + \lambda_{j_2}(r) + \dots + \lambda_{j_k}(r). \quad (11)$$

Since  $|\lambda_{\sigma(i)}(r)| > 2k \cdot |\lambda_{\sigma(i+1)}(r)|$  for every  $i \in \{0, \dots, n-1\}$ , equation (11) implies that the multisets  $\{\lambda_{i_1}(r), \lambda_{i_2}(r), \dots, \lambda_{i_k}(r)\}$  and  $\{\lambda_{j_1}(r), \lambda_{j_2}(r), \dots, \lambda_{j_k}(r)\}$  are equal. This implies that

$$\begin{aligned} \det(X \cdot I_k - A(r)) &= (X - \lambda_{j_1}(r))(X - \lambda_{j_2}(r)) \cdots (X - \lambda_{j_k}(r)) \\ &= (X - \lambda_{i_1}(r))(X - \lambda_{i_2}(r)) \cdots (X - \lambda_{i_k}(r)). \end{aligned}$$

This is precisely what we needed to prove. ■

The following is a corollary of Lemmas 1.2 and 2.5.

**Corollary 2.6** *Let  $k \geq 1$  and let  $a_{ij}(x) \in \mathbb{Z}[x]$  for all  $i, j \in \{1, \dots, k\}$ . Let  $\mathcal{R}$  be an infinite set of prime powers. Suppose that for every  $r \in \mathcal{R}$ ,  $\mathcal{P}_r = \{X_1(r), X_2(r), \dots, X_k(r)\}$  is a regular partition of  $\Delta_r$  with associated coefficients  $a_{ij}(r)$ ,  $1 \leq i, j \leq k$ . Suppose also that  $i_1, \dots, i_k$  are  $k$  not necessarily distinct elements of  $\{0, \dots, n\}$  such that  $\lambda_{i_1}(r) + \lambda_{i_2}(r) + \dots + \lambda_{i_k}(r) = a_{11}(r) + a_{22}(r) + \dots + a_{kk}(r)$  for every  $r \in \mathcal{R}$ . Then for every  $r \in \mathcal{R}$ ,  $\lambda_{i_1}(r), \lambda_{i_2}(r), \dots, \lambda_{i_k}(r)$  are all the  $k$  not necessarily distinct eigenvalues of the regular partition  $\mathcal{P}_r$  of  $\Delta_r$ .*

### 3 Class 1

(a) If  $X$  denotes the point set of the dual polar space  $\Delta$ , then  $\{X\}$  is a regular partition of  $\Delta$  with coefficient matrix  $\left[ q^e \frac{q^n - 1}{q - 1} \right]$  and associated eigenvalue  $\lambda_0 = q^e \frac{q^n - 1}{q - 1}$ .

(b) Let  $\Delta$  and  $\mathcal{P}$  be as in Class (1b). Then the coefficient matrix of the regular partition  $\mathcal{P}$  (with respect to some ordering of the elements of  $\mathcal{P}$ ) is a collinearity matrix of  $\Delta$ . So, the eigenvalues of the regular partition  $\mathcal{P}$  are the numbers  $\lambda_j$ ,  $j \in \{0, \dots, n\}$ , and the multiplicity of the eigenvalue  $\lambda_j$  of  $\mathcal{P}$  is equal to  $f_j$ .

(c) Let  $L$  be a line of size  $s + 1$  and let  $\mathcal{P} = \{X_1, X_2, \dots, X_k\}$  be a partition of  $L$ . If we put  $n_i := |X_i|$ ,  $i \in \{1, \dots, k\}$ , then  $n_1 + n_2 + \dots + n_k = s + 1$ . Clearly,  $\mathcal{P}$  is a regular partition of  $L$  with coefficient matrix

$$M = \begin{bmatrix} n_1 - 1 & n_2 & \cdots & n_k \\ n_1 & n_2 - 1 & \cdots & n_k \\ \vdots & \vdots & \ddots & \vdots \\ n_1 & n_2 & \cdots & n_k - 1 \end{bmatrix}.$$

The eigenvalues of  $\mathcal{P}$  are the roots of the polynomial  $\det(M - X \cdot I_k)$ . This polynomial is easily shown to be equal to  $(-1)^k (X - s)(X + 1)^{k-1}$ . [(i) Replace the first column of the matrix  $M - X \cdot I_k$  by the sum of all columns; (ii) Replace each entry in the first column by “1” by introducing a factor  $s - X$ ; (iii) Replace (column  $i \in \{2, \dots, k\}$ ) by (column  $i$ ) -  $n_i \times$  (column 1)]. So, the eigenvalues of the regular partition  $\mathcal{P}$  are  $s$  (multiplicity 1) and  $-1$  (multiplicity  $k - 1$ ).

### 4 Class 2

(a) Let  $\{X_0, X_1, \dots, X_k\}$  be the partition of the point-set of  $\Delta$  as described in Class (2a). Put  $n := n_1 + n_2 + \dots + n_k$ . It is straightforward to verify that  $\{X_0, X_1, \dots, X_k\}$  is a



regular partition of  $\Delta$  with associated coefficient matrix

$$\begin{bmatrix} q^e \frac{q^{n-1}-1}{q-1} - n & n_1 & n_2 & \cdots & n_k \\ q^{e+n-1} + 1 - n & q^e \frac{q^{n-1}-1}{q-1} + n_1 - 1 & n_2 & \cdots & n_k \\ q^{e+n-1} + 1 - n & n_1 & q^e \frac{q^{n-1}-1}{q-1} + n_2 - 1 & \cdots & n_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q^{e+n-1} + 1 - n & n_1 & n_2 & \cdots & q^e \frac{q^{n-1}-1}{q-1} + n_k - 1 \end{bmatrix}.$$

The eigenvalues can easily be calculated using the procedure sketched when we discussed Class (1c). The eigenvalues are  $\lambda_0 = q^e \frac{q^{n-1}-1}{q-1}$  (multiplicity 1) and  $\lambda_1 = q^e \frac{q^{n-1}-1}{q-1} - 1$  (multiplicity  $k$ ).

(b) Let  $\Delta$ ,  $X_1$  and  $X_2$  be as described in Class (2b).

Let  $M$  be a max of  $\Delta$  contained in  $X_1$ . Put  $N := |M|$ . Every point of  $M$  is contained in precisely  $\frac{q^{n-1}-1}{q-1}$  lines which are contained in  $M$ . An easy counting argument yields  $|X_1| = N + N(\alpha - \frac{q^{n-1}-1}{q-1})q^e$ . The total number of maxes which are contained in  $X_1$  is equal to  $\frac{|X_1|\beta}{N} = \left(1 + (\alpha - \frac{q^{n-1}-1}{q-1})q^e\right)\beta$ .

Now, let  $x$  be a point not contained in  $X_1$  and let  $\mu_x$  denote the number of points of  $X_1$  which are collinear with  $x$ . Counting in two different ways the number of pairs  $(y, M)$  where  $y \in X_1 \cap x^\perp$  and  $M$  a max through  $y$  contained in  $X_1$  yields  $\mu_x = 1 + (\alpha - \frac{q^{n-1}-1}{q-1})q^e$ .

By the above,  $\{X_1, X_2\}$  is a regular partition of  $\Delta$  with associated coefficient matrix

$$\begin{bmatrix} \alpha q^e & q^e \frac{q^{n-1}-1}{q-1} - \alpha q^e \\ 1 + (\alpha - \frac{q^{n-1}-1}{q-1})q^e & q^e \frac{q^{n-1}-1}{q-1} - 1 - (\alpha - \frac{q^{n-1}-1}{q-1})q^e \end{bmatrix}.$$

The eigenvalues are  $\lambda_0 = q^e \frac{q^{n-1}-1}{q-1}$  and  $\lambda_1 = q^e \frac{q^{n-1}-1}{q-1} - 1$ .

(c) Let  $\Delta$ ,  $M_1$ ,  $M_2$ ,  $X_1$ ,  $X_2$  and  $X_3$  be as described in Class (2c).

Suppose  $x \in X_1$ . Then  $x \in M_i$  for some  $i \in \{1, 2\}$ . Then  $x$  is collinear with  $q^e \frac{q^{n-1}-1}{q-1}$  other points of  $M_i$  and 1 point of  $M_{3-i}$ . So,  $x$  is collinear with  $q^e \frac{q^{n-1}-1}{q-1} + 1$  other points of  $X_1$ . If  $y$  is a point of  $X_2$  collinear with  $x$ , then  $y$  is necessarily contained in the unique line through  $x$  meeting  $M_1$  and  $M_2$ . So,  $x$  is collinear with  $q^e - 1$  points of  $X_2$  and  $q^e \frac{q^{n-1}-1}{q-1} - (q^e \frac{q^{n-1}-1}{q-1} + 1) - (q^e - 1) = q^{e+n-1} - q^e$  points of  $X_3$ .

Suppose  $x \in X_2$ . Let  $L$  denote the unique line through  $x$  meeting  $M_1$  in a point  $x_1$  and  $M_2$  in a point  $x_2$ . The point  $x$  is collinear with a unique point of  $M_1$  (namely  $x_1$ ) and a unique point of  $M_2$  (namely  $x_2$ ). So,  $x$  is collinear with precisely two points of  $X_1$ . If  $y$  is one of the  $q^e - 2$  points of  $L \setminus \{x, x_1, x_2\}$ , then  $y \in X_2$ . If  $z$  is one of the  $q^e \frac{q^{n-1}-1}{q-1}$  points of  $M_1$  at distance 1 from  $x_1$ , then the line  $z\pi_{M_2}(z)$  contains a unique point of  $X_2$  at distance 1 from  $x$ . Every point of  $X_2$  at distance 1 from  $x$  is obtained in one of the two ways described above. So,  $x$  is collinear with  $(q^e - 2) + q^e \frac{q^{n-1}-1}{q-1}$  points of  $X_2$  and  $q^e \frac{q^{n-1}-1}{q-1} - 2 - (q^e - 2) - q^e \frac{q^{n-1}-1}{q-1} = q^{e+n-1} - q^e$  points of  $X_3$ .

Suppose  $x \in X_3$ . Put  $x_1 = \pi_{M_1}(x)$  and  $x_2 = \pi_{M_2}(x)$ . If  $xx_1 = xx_2$ , then  $x_1$  and  $x_2$  are collinear and  $x$  is contained in the line  $x_1x_2$ , contradicting  $x \notin X_2$ . So,  $xx_1 \neq xx_2$  and  $Q := \langle xx_1, xx_2 \rangle$  is a quad. In fact,  $Q$  is the unique quad through  $x$  meeting  $M_1$  and  $M_2$  (in lines). Since  $xx_1 \neq xx_2$ , the points  $x_1$  and  $x_2$  are not collinear. Since  $x$  is collinear with a unique point of  $M_1$  (namely  $x_1$ ) and a unique point of  $M_2$  (namely  $x_2$ ),  $x$  is collinear with precisely 2 points of  $X_1$ . Suppose that  $y$  is a point of  $X_2$  collinear with  $x$ . If  $L_y$  denotes the unique line through  $y$  meeting  $M_1$  and  $M_2$ , then  $\langle xy, L_y \rangle$  is a quad which meets  $M_1$  and  $M_2$ . Hence,  $Q = \langle xy, L_y \rangle$  and  $y \in Q$ . Now, in the quad  $Q$  there are  $q^e + 1$  lines which meet  $M_1$  and  $M_2$ . Among these  $q^e + 1$  lines there is a line containing  $x_1$  and a line containing  $x_2$ . On each of the  $q^e - 1$  remaining lines there is a unique point of  $X_2$  collinear with  $x$ . Hence,  $x$  is collinear with  $q^e - 1$  points of  $X_2$  and  $q^e \frac{q^{n-1}-1}{q-1} - 2 - (q^e - 1) = q^e \frac{q^{n-1}-1}{q-1} - (q^e + 1)$  points of  $X_3$ .

So,  $\{X_1, X_2, X_3\}$  is a regular partition of  $\Delta$  with associated coefficient matrix:

$$\begin{bmatrix} q^e \frac{q^{n-1}-1}{q-1} + 1 & q^e - 1 & q^{e+n-1} - q^e \\ 2 & (q^e - 2) + q^e \frac{q^{n-1}-1}{q-1} & q^{e+n-1} - q^e \\ 2 & q^e - 1 & q^e \frac{q^{n-1}-1}{q-1} - (q^e + 1) \end{bmatrix}.$$

The associated eigenvalues are  $\lambda_0 = q^e \frac{q^{n-1}-1}{q-1}$  (multiplicity 1) and  $\lambda_1 = q^e \frac{q^{n-1}-1}{q-1} - 1$  (multiplicity 2).

(d) Let  $M_1, M_2, X_1, X_2$  and  $X_3$  be as in Class (2c). Then  $\{M_1, M_2, X_2, X_3\}$  is a regular partition of  $\Delta$ . The associated coefficient matrix  $M$  is readily obtained from the coefficient matrix of the regular partition  $\{X_1, X_2, X_3\}$  of  $\Delta$ . We find:

$$M = \begin{bmatrix} q^e \frac{q^{n-1}-1}{q-1} & 1 & q^e - 1 & q^{e+n-1} - q^e \\ 1 & q^e \frac{q^{n-1}-1}{q-1} & q^e - 1 & q^{e+n-1} - q^e \\ 1 & 1 & (q^e - 2) + q^e \frac{q^{n-1}-1}{q-1} & q^{e+n-1} - q^e \\ 1 & 1 & q^e - 1 & q^e \frac{q^{n-1}-1}{q-1} - (q^e + 1) \end{bmatrix}.$$

The associated eigenvalues are  $\lambda_0 = q^e \frac{q^{n-1}-1}{q-1}$  (multiplicity 1) and  $\lambda_1 = q^e \frac{q^{n-1}-1}{q-1} - 1$  (multiplicity 3).

## 5 Class 3

We continue with the notation which we introduced in Section 1.2 when we discussed Class 3. Let  $M$  denote the coefficient matrix of  $\mathcal{P}$  with respect to the ordering  $(X_1, \dots, X_k)$  and let  $M'$  denote the coefficient matrix of  $\mathcal{P}'$  with respect to the ordering  $(Y_{01}, \dots, Y_{0k}, Y_{11}, \dots, Y_{1k}, \dots, Y_{\delta 1}, \dots, Y_{\delta k})$ . We determine the entries of  $M'$  in terms of the entries of  $M$ .

Let  $x$  be a point of  $\Delta$  at distance  $i \in \{0, \dots, \delta\}$  from  $F$ . Suppose  $y = \pi_F(x)$  lies in  $X_j$  where  $j \in \{1, \dots, k\}$ . If  $y'$  is a point of  $F$  at distance  $n - \delta$  from  $y$ , then since  $y$  lies on a shortest path between  $x$  and  $y'$ , we have  $\langle x, y' \rangle = \langle x, y, y' \rangle = \langle x, F \rangle$ . So,  $F' := \langle x, F \rangle$  has

diameter  $i+n-\delta$ . Since  $d(u, v) = d(u, \pi_F(u)) + d(\pi_F(u), v) = d(u, \pi_F(u)) + n - \delta \leq n + i - \delta$  for every  $u \in F'$  and every  $v \in F$  at distance  $n - \delta$  from  $\pi_F(u)$ , we have that every point in  $F'$  has distance at most  $i$  from  $F$ .

Suppose  $L$  is a line through  $x$  containing a point  $w$  at distance  $i - 1$  from  $F$ . Then obviously  $\pi_F(w) = \pi_F(x) = y$  and  $L$  is contained in  $\langle x, y \rangle$ . The  $q^e - 1$  points of  $L$  distinct from  $x$  and  $w$  necessarily lie at distance  $i$  from  $y$  and hence also at distance  $i$  from  $F$ .

If  $w$  is a point at distance 1 from  $x$ , then  $d(\pi_F(x), \pi_F(w)) = d(y, \pi_F(w)) \leq 1$ . If  $y = \pi_F(w)$  and  $d(y, w) \leq i$ , then since the line  $xw$  contains a unique point nearest to  $y$ ,  $xw$  contains a unique point at distance  $i - 1$  from  $y$  and  $xw \subseteq \langle x, y \rangle$ .

So, we have the following:

(a) There are  $\frac{q^i - 1}{q - 1}$  points collinear with  $x$  which are contained in  $\langle x, y \rangle$  and which lie at distance  $i - 1$  from  $F$ , namely one on each line through  $x$  contained in  $\langle x, y \rangle$ . All these points lie in  $Y_{i-1, j}$  since  $y$  is the unique point of  $F$  nearest to each of these points.

(b) There are  $\frac{q^i - 1}{q - 1}(q^e - 1)$  points collinear with  $x$  which are contained in  $\langle x, y \rangle$  and which lie at distance  $i$  from  $F$ . All these points lie in  $Y_{ij}$  since  $y$  is the unique point of  $F$  nearest to each of these points.

(c) There are  $q^e \frac{q^{n-q^{i+n-\delta}}}{q-1}$  points collinear with  $x$  which are not contained in  $F'$ . All these points belong to  $Y_{i+1, \delta}$ . For, if  $u$  is one of these points, then since  $d(u, v) = d(u, x) + d(x, v)$  for every point  $v$  of  $F \subseteq F'$ , we have  $d(u, F) = d(x, F) + 1 = i + 1$  and  $\pi_F(u) = \pi_F(x) = y$ .

(d) There are  $q^e \frac{q^{i+n-\delta} - q^i}{q-1}$  points collinear with  $x$  which are contained in  $F'$ , but not in  $\langle x, y \rangle$ . All these points lie at distance  $i$  from  $F$ . If  $u$  is one of these points, then since  $u \notin \langle x, y \rangle$ , we have  $\pi_F(u) \neq y$ . So,  $\pi_F(y)$  lies at distance 1 from  $y$ . Conversely, if  $z \in X_{j'}$  lies at distance 1 from  $y$  (and hence at distance  $i + 1$  from  $x$ ), then there are precisely  $\frac{q^{i+1}-1}{q-1} - \frac{q^i-1}{q-1} = q^i$  points  $u \in X_{ij'}$  collinear with  $x$  for which  $\pi_F(u) = z$ , namely one on each line of  $\langle x, z \rangle$  through  $x$  not contained in  $\langle x, y \rangle \subseteq \langle x, z \rangle$ .

If we consider  $M'$  as a block matrix with blocks of size  $(k \times k)$ , then by the above, we have:

- $M'_{ii} = q^i \cdot M + \frac{q^i - 1}{q - 1}(q^e - 1) \cdot I_k$  for every  $i \in \{0, \dots, \delta\}$ ;
- $M'_{i, i+1} = q^e \frac{q^{n-q^{i+n-\delta}}}{q-1} \cdot I_k$  for every  $i \in \{0, \dots, \delta - 1\}$ ;
- $M'_{i, i-1} = \frac{q^i - 1}{q - 1} \cdot I_k$  for every  $i \in \{1, \dots, \delta\}$ ;
- $M'_{ij} = O_k$  for all  $i, j \in \{0, \dots, \delta\}$  with  $|i - j| \geq 2$ .

Here,  $I_k$  denotes the  $(k \times k)$  identity matrix and  $O_k$  denotes the  $(k \times k)$  zero matrix. So,  $\mathcal{P}'$  is a regular partition.

Now, for every eigenvalue  $\lambda$  of  $\tilde{F}$ , let  $N_\lambda$  denote the following  $(\delta + 1) \times (\delta + 1)$  matrix:

- $(N_\lambda)_{ii} = q^i \lambda + \frac{q^i - 1}{q - 1}(q^e - 1)$  for every  $i \in \{0, \dots, \delta\}$ ;
- $(N_\lambda)_{i, i+1} = q^e \frac{q^{n-q^{i+n-\delta}}}{q-1}$  for every  $i \in \{0, \dots, \delta - 1\}$ ;

- $(N_\lambda)_{i,i-1} = \frac{q^i-1}{q-1}$  for every  $i \in \{1, \dots, \delta\}$ ;
- $(N_\lambda)_{ij} = 0$  for all  $i, j \in \{0, \dots, \delta\}$  with  $|i - j| \geq 2$ .

By Lemma 1.3, there exists a nonsingular  $(k \times k)$ -matrix  $Q$  such that  $Q^{-1}MQ$  is a diagonal matrix. Let  $Q'$  be the  $k(\delta + 1) \times k(\delta + 1)$ -matrix which, regarded as a block matrix with blocks of size  $k \times k$ , has the following description:

- $(Q')_{ii} = Q$  for every  $i \in \{0, \dots, \delta\}$ ;
- $(Q')_{ij} = O_k$  for all  $i, j \in \{0, \dots, \delta\}$  with  $i \neq j$ .

Then we have:

- $(Q'^{-1}M'Q')_{ii} = q^i \cdot (Q^{-1}MQ) + \frac{q^i-1}{q-1}(q^e - 1) \cdot I_k$  for every  $i \in \{0, \dots, \delta\}$ ;
- $(Q'^{-1}M'Q')_{i,i+1} = q^e \frac{q^n - q^{i+n-\delta}}{q-1} \cdot I_k$  for every  $i \in \{0, \dots, \delta - 1\}$ ;
- $(Q'^{-1}M'Q')_{i,i-1} = \frac{q^i-1}{q-1} \cdot I_k$  for every  $i \in \{1, \dots, \delta\}$ ;
- $(Q'^{-1}M'Q')_{ij} = O_k$  for all  $i, j \in \{0, \dots, \delta\}$  with  $|i - j| \geq 2$ .

Let  $\lambda_{i_1}, \dots, \lambda_{i_k}$  denote the  $k$  eigenvalues of  $\mathcal{P}$  (i.e. of  $M$ ). If we apply suitable equivalent permutations to the rows and columns of the matrix  $Q'^{-1}M'Q'$ , then we find a matrix  $M''$  which, regarded as a block matrix with blocks of size  $(\delta + 1) \times (\delta + 1)$ , has the following description:

- $(M'')_{ll} = N_{\lambda_{i_l}}$  for every  $l \in \{1, \dots, k\}$ ;
- $(M'')_{ij} = O_{\delta+1}$  for all  $i, j \in \{1, \dots, k\}$  with  $i \neq j$ .

So, the eigenvalues of  $M'$  are obtained by collecting all the eigenvalues of the  $k$  matrices  $N_{\lambda_{i_1}}, \dots, N_{\lambda_{i_k}}$ .

We will now prove that for every  $j \in \{0, \dots, n - \delta\}$ , the eigenvalues of  $N_{\lambda_j}$  are  $\lambda'_j, \lambda'_{j+1}, \dots, \lambda'_{j+\delta}$ . Notice first that since  $\lambda_j$  can occur as an eigenvalue of some regular partition of  $\widetilde{F}$  (see Class (1b)), all eigenvalues of  $N_{\lambda_j}$  must be eigenvalues of  $\Delta$  by the previous paragraph. Putting  $P = N_{\lambda_j}$ , we easily find:

- $P_{ii} = \frac{q^{e+n-\delta-j+i}}{q-1} - \frac{q^{j+i}}{q-1} - \frac{q^e-1}{q-1}$  for every  $i \in \{0, \dots, \delta\}$ ;
- $P_{i,i+1} = q^e \frac{q^n - q^{i+n-\delta}}{q-1}$  for every  $i \in \{0, \dots, \delta - 1\}$ ;
- $P_{i,i-1} = \frac{q^i-1}{q-1}$  for every  $i \in \{1, \dots, \delta\}$ ;
- $P_{ij} = 0$  for all  $i, j \in \{0, \dots, \delta\}$  with  $|i - j| \geq 2$ .

For every  $i \in \{0, \dots, \delta\}$ , we have  $\lambda'_{j+i} = \frac{q^{e+n-j-i}}{q-1} - \frac{q^{j+i}}{q-1} - \frac{q^e-1}{q-1}$ ,  $i \in \{0, \dots, \delta\}$ . We clearly have  $\sum_{i=0}^{\delta} P_{ii} = \sum_{i=0}^{\delta} \lambda'_{j+i}$ . It is also straightforward to verify that  $\sum_{i=0}^{\delta} (\lambda'_{j+i})^2 - \sum_{i=0}^{\delta} (P_{ii})^2 = 2 \cdot \sum_{i=0}^{\delta-1} P_{i,i+1} P_{i+1,i} = 2 \frac{q^{e+n-\delta}}{(q-1)^2} \left( \frac{q^{2\delta+2}-1}{q^2-1} - q^\delta(\delta+1) \right)$ . By Lemmas 2.1 and 2.5, this implies that  $\lambda'_j, \lambda'_{j+1}, \dots, \lambda'_{j+\delta}$  are the eigenvalues of  $N_{\lambda_j}$ .

## 6 Class 4

(a) We continue with the notation which we introduced when we defined Class (4a). Recall that  $X_i$ ,  $i \in \mathbb{N}$ , denotes the set of all maximal singular subspaces of  $\Pi$  which intersect  $\alpha$  in an  $(m-1-i)$ -dimensional singular subspace of  $\Pi'$ . Clearly,  $X_0 \neq \emptyset$  and  $X_j = \emptyset$  if  $j \geq m+1$ .

Suppose  $x$  is a point of  $X_i$ , where  $i$  is some element of  $\{0, \dots, m\}$ . Then  $x \cap \alpha$  has dimension  $m-1-i$ . The map  $y \mapsto y \cap \alpha$  defines a bijective correspondence between the set of points  $y$  of  $X_{i-1}$  collinear with  $x$  and the set of  $(m-i)$ -dimensional singular subspaces of  $\Pi'$  containing  $x \cap \alpha$ . It follows that there are

$$a_{i,i-1} := \frac{(q^i - 1)(q^{i+e'-1} + 1)}{q - 1}$$

points of  $X_{i-1}$  collinear with  $x$ . (Notice that if  $i \geq 2$ , then  $a_{i,i-1}$  equals the total number of singular points of a polar space of rank  $i$  which is of the same type as  $\Pi'$ .)

There are two possibilities for a point  $y$  of  $X_i$  collinear with  $x$ : (i)  $y$  contains  $x \cap \alpha$ ; (ii)  $y$  does not contain  $x \cap \alpha$ .

The number of points  $y$  of  $(X_{i-1} \cup X_i) \setminus \{x\}$  collinear with  $x$  which contain  $x \cap \alpha$  is equal to  $\frac{q^{n-m+i}-1}{q-1}q^e$  since there are  $\frac{q^{n-m+i}-1}{q-1}$  hyperplanes of  $x$  containing  $x \cap \alpha$  and each of these  $\frac{q^{n-m+i}-1}{q-1}$  hyperplanes is contained in  $q^e$  maximal singular subspaces of  $\Pi$  distinct from  $x$ . Hence,  $x$  is collinear with  $\frac{q^{n-m+i}-1}{q-1}q^e - \frac{(q^i-1)(q^{i+e'-1}+1)}{q-1}$  points of  $X_i \setminus \{x\}$  which contain  $x \cap \alpha$ .

The map  $y \mapsto y \cap \alpha$  defines a bijective correspondence between the set of points of  $X_i$  collinear with  $x$  which do not contain  $x \cap \alpha$  and the set of  $(m-1-i)$ -dimensional singular subspaces  $\beta$  of  $\Pi'$  which intersect  $x \cap \alpha$  in an  $(m-2-i)$ -dimensional singular subspace of  $\Pi'$  such that  $(x \cap \alpha) \cup \beta$  is not contained in a singular subspace of  $\Pi'$ . So, the number of points of  $X_i$  collinear with  $x$  which do not contain  $x \cap \alpha$  is equal to

$$\begin{aligned} & \frac{1}{q^{m-i-1}} \left( \frac{(q^m - 1)(q^{m+e'-1} + 1)}{q - 1} - \frac{q^{m-i} - 1}{q - 1} - \frac{(q^i - 1)(q^{i+e'-1} + 1)}{q - 1} q^{m-i} \right) \\ &= q^{2i+e'} \frac{q^{m-i} - 1}{q - 1}. \end{aligned}$$

So,  $x$  is collinear with  $a_{i,i-1} = \frac{(q^i-1)(q^{i+e'-1}+1)}{q-1}$  points of  $X_{i-1}$ ,

$$a_{i,i} := \frac{q^{n-m+i} - 1}{q - 1} q^e - \frac{(q^i - 1)(q^{i+e'-1} + 1)}{q - 1} + q^{2i+e'} \frac{q^{m-i} - 1}{q - 1}$$

points of  $X_i \setminus \{x\}$ ,

$$a_{i,i+1} := q^e \frac{q^n - 1}{q - 1} - a_{i,i-1} - a_{i,i} = q^{2i+e'} \frac{(q^{m-i} - 1)(q^{n+e-m-e'-i} - 1)}{q - 1}$$

points of  $X_{i+1}$  and

$$a_{i,j} := 0$$

points of every  $X_j$  where  $j \in \{0, \dots, m\}$  with  $|i - j| \geq 2$ .

From the above coefficients, we readily observe that  $X_i \neq \emptyset$  if and only if  $i \in \{0, \dots, k\}$ , where  $k = \min(m, n + e - m - e')$ . Moreover,  $\{X_0, \dots, X_k\}$  is a regular partition of  $\Delta$ .

Notice that the eigenvalues  $\lambda_j = q^e \frac{q^{n-j}-1}{q-1} - \frac{q^j-1}{q-1}$ ,  $j \in \{0, \dots, k\}$ , satisfy the conditions (1) and (2) of Corollary 2.4.

If  $k = m$ , then  $\sum_{i=0}^k a_{ii}$  is equal to

$$\begin{aligned} & \sum_{i=0}^m q^e \frac{q^{n-m+i}-1}{q-1} - \sum_{i=0}^m \frac{q^i-1}{q-1} + \sum_{i=0}^m \frac{q^{m+i+e'} + q^{i+e'-1} - q^{2i+e'-1} - q^{2i+e'}}{q-1} \\ &= \sum_{i=0}^m q^e \frac{q^{n-i}-1}{q-1} - \sum_{i=0}^m \frac{q^i-1}{q-1} \\ &= \sum_{i=0}^k \lambda_i. \end{aligned}$$

If  $k = n + e - m - e'$ , then  $\sum_{i=0}^k a_{ii}$  is equal to

$$\begin{aligned} & \sum_{i=0}^k q^e \frac{q^{m+e'-e+i}-1}{q-1} - \sum_{i=0}^k \frac{q^i-1}{q-1} + \sum_{i=0}^k \frac{q^{e+n-m+i} + q^{i+e'-1} - q^{2i+e'-1} - q^{2i+e'}}{q-1} \\ &= \sum_{i=0}^k q^e \frac{q^{n-i}-1}{q-1} - \sum_{i=0}^k \frac{q^i-1}{q-1} \\ &= \sum_{i=0}^k \lambda_i. \end{aligned}$$

In any case, it follows from Corollaries 2.4 and 2.6, that the numbers  $\lambda_j$ ,  $j \in \{0, \dots, k\}$  are the eigenvalues of the regular partition  $\{X_0, X_1, \dots, X_k\}$  of  $\Delta$ .

(b) We will continue with the notation which we introduced when we defined Class (4b). By (a), we know that  $\{X_0^+ \cup X_0^-, X_1, \dots, X_k\}$  is a regular partition of  $\Delta$ . We denote by  $M$  the corresponding coefficient matrix. We have

$$M = \begin{bmatrix} \frac{q^{n-m}-1}{q-1}q^e + \frac{q^m-1}{q-1} & a_{01} & 0 & \cdots \\ 2 & a_{11} & a_{12} & \cdots \\ 0 & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is now straightforward to verify that also  $\{X_0^+, X_0^-, X_1, \dots, X_k\}$  is a regular partition of  $\Delta$ . Its coefficient matrix  $M'$  is easily obtained from  $M$ . We have

$$M' = \begin{bmatrix} \frac{q^{n-m}-1}{q-1}q^e & \frac{q^m-1}{q-1} & a_{01} & 0 & \cdots \\ \frac{q^m-1}{q-1} & \frac{q^{n-m}-1}{q-1}q^e & a_{01} & 0 & \cdots \\ 1 & 1 & a_{11} & a_{12} & \cdots \\ 0 & 0 & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is not so difficult to show that

$$\det(X \cdot I_{k+2} - M') = \left( X - \left( q^e \frac{q^{n-m}-1}{q-1} - \frac{q^m-1}{q-1} \right) \right) \cdot \det(X \cdot I_{k+1} - M).$$

For, consider the matrix  $X \cdot I_{k+2} - M'$ . Then the following operations can be performed on this matrix.

- (1) Subtract the second row from the first row.
- (2) Replace the first row by  $[1 \ -1 \ 0 \ 0 \ \cdots]$  by removing a factor  $X - (q^e \frac{q^{n-m}-1}{q-1} - \frac{q^m-1}{q-1})$ .
- (3) Add the first column to the second column.
- (4) Calculate the determinant of the matrix by expanding according to the first row.

So, the eigenvalues of the regular partition  $\{X_0^+, X_0^-, X_1, \dots, X_k\}$  of  $\Delta$  are the numbers  $\lambda_j$ ,  $j \in \{0, \dots, k\}$ , and the number  $\lambda_m$ .

## 7 Class 5

Let  $\Delta$  be a thick dual polar space of rank  $2n$ ,  $n \geq 1$ . A set  $X$  of points of  $\Delta$  is called an *SDPS-set* of  $\Delta$  if it satisfies the following properties:

- (1) No two points of  $X$  are collinear in  $\Delta$ .
- (2) If  $x, y \in X$  such that  $d(x, y) = 2$ , then  $X \cap \langle x, y \rangle$  is an ovoid of the quad  $\langle x, y \rangle$ .
- (3) The point-line geometry  $\tilde{\Delta}$  whose points are the elements of  $X$  and whose lines are the quads of  $\Delta$  containing at least two points of  $X$  (natural incidence) is a dual polar space of rank  $n$ .
- (4) For all  $x, y \in X$ ,  $d(x, y) = 2 \cdot \tilde{d}(x, y)$ , where  $\tilde{d}(x, y)$  denotes the distance between  $x$  and  $y$  in the dual polar space  $\tilde{\Delta}$ .
- (5) If  $x \in X$  and  $L$  is a line of  $\Delta$  through  $x$ , then  $L$  is contained in a (necessarily unique) quad of  $\Delta$  which contains at least two points of  $X$ .

SDPS-sets of dual polar spaces were introduced in De Bruyn and Vandecasteele [11], see also De Bruyn [7, Chapter 5]. If  $\Delta$  is a finite thick dual polar space of rank  $2n$  admitting an SDPS-set  $X$  with associated Sub-Dual-Polar-Space  $\tilde{\Delta}$ , then by Theorem 5.31 of [7] (whose proof is partly based on results of Pralle and Shpectorov [21]), either

- (1)  $\Delta \cong DW(4n-1, q)$  and  $\tilde{\Delta} \cong DW(2n-1, q^2)$ , or
- (2)  $\Delta \cong DQ^-(4n+1, q)$  and  $\tilde{\Delta} \cong DH(2n, q^2)$ .

For each of the two possible cases, examples of SDPS-sets are known to exist. For a proof of the following proposition, see [11] or [7, Chapter 5].

**Proposition 7.1** *Let  $X$  be an SDPS-set of a thick dual polar space  $\Delta$  of rank  $2n$ . For every point  $x$  of  $\Delta$ , put  $f(x) := d(x, X)$ . Then the following properties hold:*

- (1) *Every line  $L$  of  $\Delta$  contains a unique point  $x_L$  with smallest  $f$ -value. So,  $f(x) = f(x_L) + 1$  for every point  $x \in L \setminus \{x_L\}$ .*
- (2) *The maximal value attained by  $f$  is equal to  $n$ .*
- (3) *Every point  $x$  of  $\Delta$  is contained in a unique convex subspace  $F_x$  of diameter  $2 \cdot f(x)$  satisfying the following property: a line  $L$  through  $x$  contains a (necessarily unique) point with  $f$ -value  $f(x) - 1$  if and only if  $L$  is contained in  $F_x$ .*

Now, let  $\Delta$  be one of the dual polar spaces  $DW(4n-1, q)$ ,  $DQ^-(4n+1, q)$ , and let  $X$  be an SDPS-set of  $\Delta$ . For every  $i \in \{0, \dots, n\}$ , let  $X_i$  denote the set of points of  $\Delta$  at distance  $i$  from  $X$ . Then by Proposition 7.1,  $\mathcal{P} = \{X_0, X_1, \dots, X_n\}$  is a regular partition of  $\Delta$ . Also, by Proposition 7.1, the parameters  $a_{ij}$ ,  $i, j \in \{0, \dots, n\}$ , of the regular partition are easily calculated. We find

- $a_{ii} = \frac{q^{2i}-1}{q-1}(q^e - 1) = \frac{q^{e+2i}}{q-1} - \frac{q^{2i}}{q-1} - \frac{q^e-1}{q-1}$  for every  $i \in \{0, \dots, n\}$ ;
- $a_{i,i+1} = \frac{q^{2n-q^{2i}}}{q-1}q^e$  for every  $i \in \{0, \dots, n-1\}$ ;
- $a_{i,i-1} = \frac{q^{2i}-1}{q-1}$  for every  $i \in \{1, \dots, n\}$ ;
- $a_{ij} = 0$  for all  $i, j \in \{0, \dots, n\}$  with  $|i-j| \geq 2$ .

Here,  $e = 1$  if  $\Delta = DW(4n-1, q)$  and  $e = 2$  if  $\Delta = DQ^-(4n+1, q)$ . Recall that  $\lambda_{2i} = q^e \frac{q^{2n-2i}-1}{q-1} - \frac{q^{2i}-1}{q-1} = \frac{q^{e+2n-2i}}{q-1} - \frac{q^{2i}}{q-1} - \frac{q^e-1}{q-1}$  for every  $i \in \{0, \dots, n\}$ . Obviously, we have that  $\sum_{i=0}^n \lambda_{2i} = \sum_{i=0}^n a_{ii}$ . It is straightforward to verify that  $\sum_{i=0}^n (\lambda_{2i})^2 - \sum_{i=0}^n (a_{ii})^2 = 2 \cdot \sum_{i=0}^{n-1} a_{i,i+1} \cdot a_{i+1,i} = \frac{2q^e}{(q-1)^2} \left( \frac{q^{4n+4}-1}{q^4-1} - q^{2n}(n+1) \right)$ . Corollary 2.2 now implies that the numbers  $\lambda_{2i}$ ,  $i \in \{0, \dots, n\}$ , are the eigenvalues of  $\mathcal{P}$ .

## 8 Class 6

We take the following results from De Bruyn [8].

**Proposition 8.1** *Let  $Y$  be a set of points of  $DH(2n-1, q^2)$ ,  $n \geq 2$ , such that  $\tilde{Y} \cong DW(2n-1, q)$  is isometrically embedded into  $DH(2n-1, q^2)$ . Then:*

- (1) *The maximal distance between a point of  $DH(2n-1, q^2)$  and the set  $Y$  is equal to  $\lfloor \frac{n}{2} \rfloor$ .*
- (2) *If  $x$  is a point of  $DH(2n-1, q^2)$  at distance  $\delta \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$  from  $Y$ , then  $\Gamma_\delta(x) \cap Y$  is an SDPS-set of a convex subspace of diameter  $2\delta$  of  $\tilde{Y}$ . Moreover,  $d(x, y) = \delta + d(\Gamma_\delta(x) \cap Y, y)$  for every point  $y \in Y$ .*



(3) If  $n$  is even, then the set of points of  $DH(2n-1, q^2)$  at distance at most  $\frac{n}{2} - 1$  from  $Y$  is a hyperplane of  $DH(2n-1, q^2)$ .

Now, let  $X$  be a set of points of  $DH(2n-1, q^2)$  such that  $\tilde{X} \cong DW(2n-1, q)$  is isometrically embedded into  $DH(2n-1, q^2)$ . For every convex subspace  $F$  of  $\tilde{X}$ , let  $\overline{F}$  denote the unique convex subspace of  $DH(2n-1, q^2)$  containing  $F$  and having the same diameter as  $F$ . Put  $m := \lfloor \frac{n}{2} \rfloor$  and let  $X_i$ ,  $i \in \{0, \dots, m\}$ , denote the set of points of  $DH(2n-1, q^2)$  at distance  $i$  from  $X$ . Then by Proposition 8.1(1),  $\{X_0, \dots, X_m\}$  is a partition of the point set of  $DH(2n-1, q^2)$ .

We will now prove that  $\{X_0, \dots, X_m\}$  is a regular partition of  $DH(2n-1, q^2)$ . Let  $x$  be a point of  $DH(2n-1, q^2)$  at distance  $i$  from  $X$ . Then  $A_x := \Gamma_i(x) \cap X$  is an SDPS-set of a convex subspace  $F_x$  of diameter  $2i$  of  $\tilde{X}$ . The maximal distance between two points of  $A_x$  is equal to  $2i$ . So,  $x$  lies on a shortest path between two points of  $A_x \subseteq F_x$ . Hence,  $x \in \overline{F_x}$ .

Suppose  $u$  is a point of  $DH(2n-1, q^2)$  collinear with  $x$  at distance  $i-1$  from  $X$ . Then  $\Gamma_{i-1}(u) \cap X \subseteq \Gamma_i(x) \cap X \subseteq F_x$ . So,  $u$  has also distance  $i-1$  from  $F_x$ . Moreover,  $u$  is contained in  $\overline{F_x}$  since it lies on a shortest path between  $x \in \overline{F_x}$  and a point of  $\Gamma_i(x) \cap X \subseteq F_x$ .

By Proposition 8.1(3) applied to the isometric embedding of  $\tilde{F_x}$  into  $\widetilde{\overline{F_x}}$ , we see that every line of  $\overline{F_x}$  through  $x$  contains a unique point at distance  $i-1$  from  $F_x$ . By the previous paragraph, this point is also the unique point on that line at distance  $i-1$  from  $X$ .

By the previous two paragraphs we know that  $x$  is collinear with precisely  $\frac{q^{4i}-1}{q^2-1}$  points of  $X_{i-1}$  and  $\frac{q^{4i}-1}{q^2-1}(q-1) = \frac{q^{4i}-1}{q+1}$  points of  $X_i \cap \overline{F_x}$  distinct from  $x$ .

Now, suppose that  $F$  is one of the  $\frac{q^{n-2i}-1}{q-1}$  convex subspaces of diameter  $2i+1$  of  $\tilde{X}$  containing  $F_x$ . If  $L$  is one of the  $q^{4i}$  lines through  $x$  contained in  $\overline{F}$  but not in  $\overline{F_x}$ , then by Proposition 8.1(1) and the discussion above,  $L \setminus \{x\}$  is completely contained in  $X_i$ . The number of points of  $X_i$  collinear with  $x$  which arise in this way is equal to  $\frac{q^{n-2i}-1}{q-1}q^{4i+1}$ .

Now, suppose that  $L$  is a line through  $x$  not contained in  $\overline{F_x}$  which contains a point  $u \neq x$  at distance  $i$  from  $X$ . Then every point of  $\Gamma_i(x) \cap X \subseteq \overline{F_x}$  has distance  $i+1$  from  $u$  and every point of  $F_x \subseteq \overline{F_x}$  has distance at least  $i+1$  from  $u$ . Let  $v$  be an arbitrary point of  $\Gamma_i(u) \cap X$ . Then since  $v \notin F_x$  and  $d(x, v) \leq d(x, u) + d(u, v) = i+1$ ,  $v$  has distance  $i+1$  from  $x$ . By Proposition 8.1(2),  $v$  is collinear with some point of  $\Gamma_i(x) \cap X \subseteq F_x$ . So,  $F := \langle F_x, v \rangle$  is a convex subspace of diameter  $2i+1$  of  $\tilde{X}$ . Since  $u$  is contained on a shortest path between  $x \in \overline{F}$  and  $v \in \overline{F}$ , we have  $L = xu \subseteq \overline{F}$ .

By the above discussion,  $x$  is collinear with  $\frac{q^{4i}-1}{q+1} + \frac{q^{n-2i}-1}{q-1}q^{4i+1} = \frac{q^{n+2i+1}}{q-1} - \frac{q^2+1}{q^2-1}q^{4i} - \frac{1}{q+1}$  other points of  $X_i$  and  $q\frac{q^{2n}-1}{q^2-1} - \frac{q^{4i}-1}{q^2-1} - (\frac{q^{n+2i+1}}{q-1} - \frac{q^2+1}{q^2-1}q^{4i} - \frac{1}{q+1}) = \frac{q^{4i+2}(q^{n-2i}-1)(q^{n-2i}-1)}{q^2-1}$  points of  $X_{i+1}$ .

Summarizing, we have that  $\{X_0, X_1, \dots, X_m\}$  is a regular partition of  $DH(2n-1, q^2)$  with coefficients:

- $a_{ii} = \frac{q^{n+2i+1}}{q-1} - \frac{q^2+1}{q^2-1}q^{4i} - \frac{1}{q+1}$  for every  $i \in \{0, \dots, m\}$ ;
- $a_{i,i-1} = \frac{q^{4i}-1}{q^2-1}$  for every  $i \in \{1, \dots, m\}$ ;
- $a_{i,i+1} = \frac{q^{4i+2}(q^{n-2i-1}-1)(q^{n-2i}-1)}{q^2-1}$  for every  $i \in \{0, \dots, m-1\}$ ;
- $a_{ij} = 0$  for all  $i, j \in \{0, \dots, m\}$  with  $|i-j| \geq 2$ .

Recall that  $\lambda_i = q \frac{q^{2n-2i}-1}{q^2-1} - \frac{q^{2i}-1}{q^2-1}$  for every  $i \in \{0, \dots, m\}$ . If  $n$  is even (so,  $m = \frac{n}{2}$ ), then one calculates that  $\sum_{i=0}^m \lambda_i = \sum_{i=0}^m a_{ii} = \frac{(q^{n+1}-1)(q^{n+2}-1)}{(q^2-1)^2} - \frac{n+2}{2(q+1)}$ . If  $n$  is odd (so,  $m = \frac{n-1}{2}$ ), then one calculates that  $\sum_{i=0}^m \lambda_i = \sum_{i=0}^m a_{ii} = \frac{(q^{n+1}-1)(q^{n+2}-1)}{(q^2-1)^2} - \frac{n+1}{2(q+1)}$ . Corollary 2.6 implies that the numbers  $\lambda_j$ ,  $j \in \{0, \dots, m\}$ , are the eigenvalues of the regular partition  $\{X_0, X_1, \dots, X_m\}$  of  $DH(2n-1, q^2)$ .

## 9 Class 7

(a) By Shult [22] (see also Pralle [20]), there exists a hyperplane  $X_1$  of  $DQ(6, q)$  such that  $\widetilde{X_1}$  is isomorphic to the split-Cayley hexagon  $H(q)$ . Let  $X_2$  denote the complement of  $X_1$  in  $DQ(6, q)$ . Recall that  $H(q)$  has  $(q^3+1)(q^2+q+1)$  points and that every point of  $H(q)$  is incident with precisely  $q+1$  lines. So,  $|X_1| = (q^3+1)(q^2+q+1)$  and  $|X_2| = (q+1)(q^2+1)(q^3+1) - |X_1| = (q^3+1)q^3$ . Every point of  $X_1$  is collinear with  $q(q+1)$  other points of  $X_1$  and  $q^3$  points of  $X_2$ . Every point  $x \in X_2$  is collinear with  $q^2+q+1$  points of  $X_1$  (one on each line through  $x$ ) and  $q^3-1$  points of  $X_2$ . So,  $\{X_1, X_2\}$  is a regular partition of  $DQ(6, q)$  with coefficient matrix

$$\begin{bmatrix} (q+1)q & q^3 \\ q^2+q+1 & q^3-1 \end{bmatrix}.$$

The eigenvalues of the regular partition are  $\lambda_0 = q^3 + q^2 + q$  and  $\lambda_2 = -1$ .

(b) Let  $Q^-(7, q)$ ,  $\pi$ ,  $Q(6, q)$ ,  $X_1$ ,  $X_2$ ,  $X_3$  and  $X_4$  be as described in Class (7b).

Let  $x$  be a point of  $X_1$ . Then as in (a) above,  $x$  is collinear with  $(q+1)q$  other points of  $X_1$  and  $q^3$  points of  $X_2$ . There are  $q+1$  singular lines of  $Q(6, q)$  contained in the singular plane  $x$  with the property that every singular plane of  $Q(6, q)$  through each of them belongs to  $X_1$ . Each such line is contained in precisely  $q^2-q$  singular planes belonging to  $X_3$ . So,  $x$  is collinear with  $(q+1)(q^2-q)$  points of  $X_3$  and  $q^2(q^2+q+1) - (q+1)q - q^3 - (q+1)(q^2-q) = q^3(q-1)$  points of  $X_4$ .

Let  $x$  be a point of  $X_2$ . Then as in (a) above,  $x$  is collinear with  $q^2+q+1$  points of  $X_1$  and  $q^3-1$  points of  $X_2$ . Clearly,  $x$  cannot be collinear with points of  $X_3$ . So,  $x$  is collinear with  $q^2(q^2+q+1) - (q^2+q+1) - (q^3-1) = q^4 - q$  points of  $X_4$ .

Let  $x$  be a point of  $X_3$ . Then  $x \cap \pi$  is a singular line of  $Q(6, q)$  which is contained in  $q+1$  singular planes of  $X_1$  and 0 singular planes of  $X_2$ . There are  $q^2-q-1$  singular planes of  $X_3 \setminus \{x\}$  through  $x \cap \pi$ . We show that these are all the singular planes of  $X_3 \setminus \{x\}$

which are collinear with  $x \in X_3$ . Suppose to the contrary that  $y$  is a singular plane of  $X_3 \setminus \{x\}$  collinear with  $x$  which does not contain the line  $x \cap \pi$ . Then  $x \cap y$  and  $y \cap \pi$  are two singular lines of  $Q^-(7, q)$  which intersect  $x \cap \pi$  in a point  $p$ . The singular lines  $x \cap \pi$  and  $y \cap \pi$  of  $Q^-(7, q)$  respectively correspond to lines  $L_1$  and  $L_2$  of  $DQ(6, q)$  which are contained in the quad  $Q$  of  $DQ(6, q)$  corresponding to the point  $p$ . Since  $x, y \in X_3$ , the lines  $L_1$  and  $L_2$  are contained in the hexagonal hyperplane of  $DQ(6, q)$  defined by  $X_1$ . Now, if a hexagonal hyperplane of  $DQ(6, q)$  contains two distinct lines which are contained in a quad, then these two lines necessarily meet in a point. Applying this fact here, we see that  $\langle x \cap \pi, y \cap \pi \rangle$  must be a singular plane of  $Q^-(7, q)$ . But this is impossible, since  $y$  is the unique singular plane through  $y \cap \pi$  which meets the singular plane  $x$  in a singular line. It follows that the point  $x$  is collinear with precisely  $q^2 - q - 1$  other points of  $X_3$ . So,  $x$  is collinear with  $q^2(q^2 + q + 1) - (q + 1) - (q^2 - q - 1) = q^3(q + 1)$  points of  $X_4$ .

Let  $x$  be a point of  $X_4$ . The singular line  $x \cap \pi$  of  $Q(6, q)$  is contained in  $q + 1$  singular planes of  $Q(6, q)$ , precisely one of which belongs to  $X_1$ . So,  $x$  is collinear with 1 point of  $X_1$  and  $q$  points of  $X_2$ . Since  $X_1 \cup X_3$  is a hyperplane,  $x$  is collinear with precisely  $q^2 + q + 1$  points of  $X_1 \cup X_3$  (one on each line through  $x$ ). It follows that  $x$  is collinear with  $q^2 + q$  points of  $X_3$  and  $q^2(q^2 + q + 1) - 1 - q - (q^2 + q) = q^4 + q^3 - 2q - 1$  points of  $X_4$ .

So,  $\{X_1, X_2, X_3, X_4\}$  is a regular partition of  $DQ^-(7, q)$  with coefficient matrix

$$\begin{bmatrix} (q+1)q & q^3 & (q+1)(q^2-q) & q^2(q^2-q) \\ q^2+q+1 & q^3-1 & 0 & q^4-q \\ q+1 & 0 & q^2-q-1 & q^4+q^3 \\ 1 & q & q^2+q & q^4+q^3-2q-1 \end{bmatrix}.$$

The associated eigenvalues are  $\lambda_1 = q^4 + q^3 + q^2$ ,  $\lambda_2 = q^3 + q^2 - 1$ ,  $\lambda_3 = q^2 - q - 1$  and  $\lambda_4 = -q^2 - q - 1$ .

(c) Let  $Q(8, q)$ ,  $\pi$ ,  $Q^+(7, q)$ ,  $\mathcal{M}^+$ ,  $\mathcal{M}^-$ ,  $X_1$ ,  $X_2$ ,  $X_3$ ,  $X_4$  and  $X_5$  be as described in Class (7c). Recall that by Cardinali, De Bruyn and Pasini [4],  $H := X_1 \cup X_2 \cup X_3$  is a hyperplane of the dual polar space  $\Delta = DQ(8, q)$  associated with  $Q(8, q)$ . If  $F$  is a hex of  $\Delta$ , then by [4], one of the following two possibilities occurs:

(i) There exists a point  $x \in F$  such that  $F \cap H$  consists of all points of  $F$  at distance at most 2 from  $x$ . If this case occurs, then  $F$  is called a *singular hex*. The point  $x$  is called the *deepest point* of  $F$ .

(ii)  $F \cap H$  is a hexagonal hyperplane of  $\tilde{F}$ . If this case occurs, then  $F$  is called a *hexagonal hex*.

**Lemma 9.1** (i) *If  $x$  is a point of  $H$ , then either every line through  $x$  is contained in  $H$  or there exists a unique hex  $F$  through  $x$  such that the lines through  $x$  contained in  $H$  are precisely the lines through  $x$  contained in  $F$ .*

(ii) *The singular hexes of  $\Delta$  are precisely the hexes of  $\Delta$  which correspond to the points of  $Q(8, q)$  which are contained in  $Q^+(7, q)$ .*

(iii) If  $F$  is a singular hex with deepest point  $x$ , then  $x \in X_1 = \mathcal{M}^+$ . Conversely, every point of  $X_1 = \mathcal{M}^+$  is the deepest point of a unique singular hex.

(iv) Let  $x \in X_1 = \mathcal{M}^+$ . Then the lines through  $x$  which are contained in  $H$  are precisely the lines through  $x$  which are contained in the unique singular hex which has  $x$  as deepest point.

**Proof.** Claims (i), (ii) and (iii) have been proved in Cardinali, De Bruyn and Pasini [4]. We will now also prove Claim (iv). Let  $x \in X_1 = \mathcal{M}^+$  and let  $F$  denote the unique singular hex through  $x$  which has  $x$  as deepest point. Then every line through  $x$  which is contained in  $F$  is also contained in  $H$ . Let  $L$  be an arbitrary line through  $x$  not contained in  $F$  and let  $F'$  denote an arbitrary hex through  $L$ . The point of  $Q(8, q)$  corresponding to  $F'$  is contained in the maximal singular subspace  $x \in \mathcal{M}^+$ . Hence, by Lemma 9.1(ii),  $F'$  is a singular hex. Since the quad  $F \cap F'$  is contained in  $H$ , the deepest point  $x'$  of  $F'$  is contained in  $F \cap F'$ . Since  $x, x' \in \mathcal{M}^+$  and  $x \neq x'$ , we necessarily have  $d(x, x') = 2$ . This implies that  $L \subseteq F'$  is not contained in  $H$  since  $L \setminus \{x\} \subseteq \Delta_3(x')$ . ■

Let  $x$  be a point of  $X_1 = \mathcal{M}^+$ . Clearly, no other point of  $\mathcal{M}^+$  is collinear with  $x$ . Let  $F$  denote the unique singular hex for which  $x$  is the deepest point. If  $L$  is one of the  $q^2 + q + 1$  lines through  $x$  contained in  $F$ , then  $L$  contains a unique point of  $X_2$  and  $q - 1$  points of  $X_3$ . If  $L$  is one of the  $q^3$  lines through  $x$  not contained in  $F$ , then by Lemma 9.1(iv)  $L$  contains a unique point of  $X_4$  and  $(q - 1)$  points of  $X_5$ . It follows that  $x$  is collinear with 0 other points of  $X_1$ ,  $q^2 + q + 1$  points of  $X_2$ ,  $(q^2 + q + 1)(q - 1) = q^3 - 1$  points of  $X_3$ ,  $q^3$  points of  $X_4$  and  $q^3(q - 1)$  points of  $X_5$ .

Let  $x$  be a point of  $X_2$ . Then every line through  $x$  is contained in  $H$ . Each such line contains a unique point of  $X_1 = \mathcal{M}^+$  and  $q - 1$  points of  $X_3$ . So,  $x$  is collinear with  $q^3 + q^2 + q + 1$  points of  $X_1$ , 0 points of  $X_2$ ,  $(q^3 + q^2 + q + 1)(q - 1) = q^4 - 1$  points of  $X_3$ , 0 points of  $X_4$  and 0 points of  $X_5$ .

Let  $x$  be a point of  $X_3$ . Then  $x$  is collinear with a unique point of  $X_1 = \mathcal{M}^+$ , a unique point of  $X_2 \subseteq \mathcal{M}^-$  and no point of  $X_4 \subseteq \mathcal{M}^-$ . If every line through  $x$  is contained in  $H$ , then every hex through  $x$  is singular, which is impossible by Lemma 9.1(ii) and the fact that the maximal singular subspace  $x$  is not contained in  $Q^+(7, q)$ . By Lemma 9.1(i), precisely  $q^2 + q + 1$  lines through  $x$  are contained in  $H$ . So,  $x$  is collinear with  $q(q^2 + q + 1) - 1 - 1 = q^3 + q^2 + q - 2$  points of  $X_3$  and  $q^4 + q^3 + q^2 + q - 1 - 1 - (q^3 + q^2 + q - 2) = q^4$  points of  $X_4$ .

Let  $x$  be a point of  $X_4$ . Every line through  $x$  contains a unique point of  $X_1 = \mathcal{M}^+$  and  $q - 1$  points of  $X_5$ . So,  $x$  is collinear with  $q^3 + q^2 + q + 1$  points of  $X_1$ , 0 other points of  $X_2 \cup X_3 \cup X_4$  and  $(q - 1)(q^3 + q^2 + q + 1) = q^4 - 1$  points of  $X_5$ .

Let  $x$  be a point of  $X_5$ . Then  $x$  is collinear with a unique point of  $\mathcal{M}^+$ , a unique point of  $X_4 \subseteq \mathcal{M}^-$  and no point of  $X_2 \subseteq \mathcal{M}^-$ . Since  $H$  is a hyperplane, the point  $x$  is collinear with precisely  $q^3 + q^2 + q + 1$  points of  $H = X_1 \cup X_2 \cup X_3$ . It follows that  $x$  is collinear with a unique point of  $X_1$ , 0 points of  $X_2$ ,  $q^3 + q^2 + q + 1 - 1 = q^3 + q^2 + q$  points of  $X_3$ , 1 point of  $X_4$  and  $q^4 + q^3 + q^2 + q - 1 - (q^3 + q^2 + q) - 1 = q^4 - 2$  points of  $X_4$ .

So,  $\{X_1, X_2, X_3, X_4, X_5\}$  is a regular partition of  $\Delta$  with coefficient matrix

$$\begin{bmatrix} 0 & q^2 + q + 1 & q^3 - 1 & q^3 & q^4 - q^3 \\ q^3 + q^2 + q + 1 & 0 & q^4 - 1 & 0 & 0 \\ 1 & 1 & q^3 + q^2 + q - 2 & 0 & q^4 \\ q^3 + q^2 + q + 1 & 0 & 0 & 0 & q^4 - 1 \\ 1 & 0 & q^3 + q^2 + q & 1 & q^4 - 2 \end{bmatrix}.$$

The associated eigenvalues are  $\lambda_0 = q^4 + q^3 + q^2 + q$ ,  $\lambda_1 = q^3 + q^2 + q - 1$ ,  $\lambda_2 = q^2 - 1$ ,  $\lambda_3 = -q^2 - 1$  and  $\lambda_4 = -q^3 - q^2 - q - 1$ .

(d) Let  $X_1$ ,  $X_2$  and  $X_3$  be sets of points of  $DH(5, q^2)$  as described in Class (7d). By De Bruyn and Pralle [10], there exist two partitions of  $X_1$  in  $q^3 + 1$  quads. Two quads belonging to distinct partitions intersect in a line. Also by [10], through each point of  $X_1$  there are  $q^3 + q^2 + 1$  lines which are contained in  $X_1 \cup X_2$ ,  $2q^2 + 1$  of these lines are contained in  $X_1$  and the remaining  $q^3 - q^2$  lines contain  $q$  points of  $X_3$ . Through each point of  $X_2$  there are  $q^3 + 1$  lines which are contained in  $X_1 \cup X_2$  and each of these lines contains a unique point of  $X_1$ . Notice also that since  $X_1 \cup X_2$  is a hyperplane of  $DH(5, q^2)$ , every point  $x \in X_3$  is collinear with  $q^4 + q^2 + 1$  points of  $X_1 \cup X_2$  and  $q^3 + 1$  of these points are contained in  $X_1$  (recall that  $X_1$  has a partition in  $q^3 + 1$  quads and that each such quad contains a unique point collinear with  $x$ ).

It follows that  $\{X_1, X_2, X_3\}$  is a regular partition of  $DH(5, q^2)$  with associated coefficient matrix

$$\begin{bmatrix} 2q^3 + q & q^4 - q^3 & q^5 - q^4 \\ q^3 + 1 & (q^3 + 1)(q - 1) & q^3(q^2 - q + 1) \\ q^3 + 1 & q^4 - q^3 + q^2 & (q - 1)(q^4 + q^2 + 1) \end{bmatrix}.$$

The associated eigenvalues are  $\lambda_0 = q^5 + q^3 + q$ ,  $\lambda_1 = q^3 + q - 1$  and  $\lambda_2 = -q^2 + q - 1$ .

## 10 Class 8

(a) Let  $X_1$  be a subspace of  $DW(5, q)$  such that  $\widetilde{X_1}$  is a  $(q+1) \times (q+1) \times (q+1)$ -cube which is isometrically embedded into  $DW(5, q)$ . So,  $|X_1| = (q+1)^3$ . Every quad  $Q$  of  $\widetilde{X_1}$  is contained in a unique quad  $\overline{Q}$  of  $DW(5, q)$ . This quad  $\overline{Q}$  contains  $|\overline{Q}| - |Q| = (q+1)(q^2 - q)$  points of  $DW(5, q)$  not contained in  $X_1$ . Put  $X_2 = \bigcup_Q (\overline{Q} \setminus Q)$ , where the union ranges over all quads  $Q$  of  $\widetilde{X_1}$ . We have  $|X_2| = 3(q+1) \cdot (q+1)(q^2 - q) = 3q(q-1)(q+1)^2$ . A point  $x \in X_1$  is contained in  $q^2 + q + 1$  lines of  $DW(5, q)$  and three quads  $Q_1, Q_2, Q_3$  of  $\widetilde{X_1}$ . One easily verifies that there are  $(q-1)^2$  lines through  $x$  not contained in  $\overline{Q_1} \cup \overline{Q_2} \cup \overline{Q_3}$ . Now, let  $X_3$  denote the set of points of  $DW(5, q)$  which are collinear with a unique point of  $X_1$ . Then by the above  $|X_3| = |X_1| \cdot (q-1)^2 \cdot q = q(q-1)^2(q+1)^3$ . Since  $|X_1| + |X_2| + |X_3| = (q+1)(q^2+1)(q^3+1)$  we have counted all the points of  $DW(5, q)$ . We will now prove that  $\{X_1, X_2, X_3\}$  is a regular partition of  $DW(5, q)$ .

Let  $x$  be a point of  $X_1$ . There are 3 lines  $L$  through  $x$  such that  $L \setminus \{x\}$  is contained in  $X_1$ ,  $3(q-1)$  lines  $L$  through  $x$  such that  $L \setminus \{x\}$  is contained in  $X_2$  and  $(q-1)^2$  lines

$L$  through  $x$  such that  $L \setminus \{x\}$  is contained in  $X_3$ . It follows that  $x$  is collinear with  $3q$  other points of  $X_1$ ,  $3q(q-1)$  points of  $X_2$  and  $q(q-1)^2$  points of  $X_3$ .

Let  $x$  be a point of  $X_2$ . Then there exists a unique quad  $Q$  of  $\widetilde{X_1}$  such that  $x \in \overline{Q} \setminus Q$ . The subgrid  $Q$  of  $\overline{Q}$  contains  $q+1$  points collinear with  $x$  and these are all the points of  $X_1$  collinear with  $x$ . The points of  $X_2$  which are collinear with  $x$  are either contained in  $\overline{Q} \setminus Q$  or contained in one of the quads  $\overline{R}$ , where  $R$  is one of the  $q$  quads of  $\widetilde{X_1}$  disjoint from  $Q$ . Each such quad  $\overline{R}$  is disjoint from  $\overline{Q}$  and contains a unique point (of  $X_2$ ) collinear with  $x$ . So,  $x$  is collinear with  $(q^2-1) + q = q^2 + q - 1$  points of  $X_2$ . It follows that  $x$  is collinear with  $q(q^2 + q + 1) - (q+1) - (q^2 + q - 1) = q^3 - q$  points of  $X_3$ .

Let  $x$  be a point of  $X_3$ . Then there exists a unique point  $y$  of  $X_1$  collinear with  $x$ . Each point of  $X_2$  collinear with  $x$  is contained in one of the quads  $\overline{Q}$ , where  $Q$  is one of the  $3q$  quads of  $\widetilde{X_1}$  not containing  $y$ . Each such quad  $\overline{Q}$  contains a unique point (of  $X_2$ ) collinear with  $x$ . It follows that  $x$  is collinear with  $3q$  points of  $X_2$ . So,  $x$  must be collinear with  $q(q^2 + q + 1) - 1 - 3q = q^3 + q^2 - 2q - 1$  points of  $X_3$ .

So,  $\{X_1, X_2, X_3\}$  is a regular partition with associated coefficient matrix

$$\begin{bmatrix} 3q & 3q(q-1) & q(q-1)^2 \\ q+1 & q^2+q-1 & q^3-q \\ 1 & 3q & q^3+q^2-2q-1 \end{bmatrix}.$$

The associated eigenvalues are  $\lambda_0 = q^3 + q^2 + q$ ,  $\lambda_1 = q^2 + q - 1$  and  $\lambda_2 = -1$ .

(b) Suppose  $X_1$  is a subspace of  $DH(5, q^2)$  such that  $\widetilde{X_1}$  is a  $(q+1) \times (q+1) \times (q+1)$ -cube which is isometrically embedded into  $DH(5, q^2)$ . So,  $|X_1| = (q+1)^3$ . Every quad  $Q$  of  $\widetilde{X_1}$  is contained in a unique quad  $\overline{Q}$  of  $DH(5, q^2)$ . This quad  $\overline{Q}$  contains  $(q+1)(q^3-q)$  points which are not contained in  $X_1$ . Put  $X_2 = \bigcup_Q (\overline{Q} \setminus Q)$ , where the union ranges over all quads  $Q$  of  $\widetilde{X_1}$ . We have  $|X_2| = 3(q+1) \cdot (q+1)(q^3-q) = 3q(q-1)(q+1)^3$ . A point of  $x \in X_1$  is contained in  $q^4 + q^2 + 1$  lines of  $DH(5, q^2)$  and three quads  $Q_1, Q_2, Q_3$  of  $\widetilde{X_1}$ . One easily verifies that there are  $(q^2-1)^2$  lines through  $x$  not contained in  $\overline{Q_1} \cup \overline{Q_2} \cup \overline{Q_3}$ . Now, let  $X_3$  denote the set of points of  $DH(5, q^2)$  not contained in  $X_1 \cup X_2$  which are collinear with a (necessarily unique) point of  $X_1$ . Then by the above,  $|X_3| = |X_1| \cdot (q^2-1)^2 \cdot q = q(q-1)^2(q+1)^5$ . Let  $X_4$  denote the set of points of  $DH(5, q^2)$  not contained in  $X_1 \cup X_2 \cup X_3$ . Then  $|X_4| = (q+1)(q^3+1)(q^5+1) - |X_1| - |X_2| - |X_3| = q^3(q-1)^3(q+1)^3$ . We will now prove that  $\{X_1, X_2, X_3, X_4\}$  is a regular partition of  $DH(5, q^2)$ .

Let  $x$  be a point of  $X_1$ . There are 3 lines  $L$  through  $x$  such that  $L \setminus \{x\}$  is contained in  $X_1$ ,  $3(q^2-1)$  lines  $L$  through  $x$  such that  $L \setminus \{x\}$  is contained in  $X_2$  and  $(q^2-1)^2$  lines  $L$  through  $x$  such that  $L \setminus \{x\}$  is contained in  $X_3$ . It follows that  $x$  is collinear with  $3q$  other points of  $X_1$ ,  $3q(q^2-1)$  points of  $X_2$ ,  $q(q^2-1)^2$  points of  $X_3$  and 0 points of  $X_4$ .

Let  $x$  be a point of  $X_2$ . Then there exists a unique quad  $Q$  of  $\widetilde{X_1}$  such that  $x \in \overline{Q} \setminus Q$ . The subgrid  $Q$  of  $\widetilde{X_1}$  contains  $q+1$  points collinear with  $x$  and these are all the points of  $X_1$  collinear with  $x$ . The points of  $X_2$  which are collinear with  $x$  are either contained in  $\overline{Q} \setminus Q$  or in a quad  $\overline{R}$  where  $R$  is one of the  $q$  quads of  $\widetilde{X_1}$  which are disjoint from  $Q$ . Each such quad  $\overline{R}$  is disjoint from  $\overline{Q}$  and contains a unique point (of  $X_2$ ) collinear

with  $x$ . It follows that  $x$  is collinear with  $[q(q^2 + 1) - (q + 1)] + q = q^3 + q - 1$  points of  $X_2$ . There are  $q$  quads  $R$  of  $\widetilde{X}_1$  which are disjoint from  $Q$  and each of these quads contains  $q + 1$  points at distance 2 from  $x$ . If  $y_R$  is one of these points, then  $x$  and  $y_R$  have  $q^2 - 1$  neighbors which are not contained in  $\overline{Q} \cup \overline{R}$ . All these neighbors belong to  $X_3$ . Conversely, every point of  $X_3$  collinear with  $x$  is obtained in this way. It follows that  $x$  is collinear with  $q(q + 1)(q^2 - 1) = q(q - 1)(q + 1)^2$  points of  $X_3$ . Hence,  $x$  is collinear with  $q^5 + q^3 + q - (q + 1) - (q^3 + q - 1) - q(q - 1)(q + 1)^2 = q^2(q - 1)^2(q + 1)$  points of  $X_4$ .

Let  $x$  be a point of  $X_3$ . Then  $x$  is collinear with a unique point  $y$  of  $X_1$ . The points of  $X_2$  which are collinear with  $x$  are contained in a quad  $\overline{Q}$  where  $Q$  is one of the  $3q$  quads of  $\widetilde{X}_1$  not containing  $y$ . Each such quad  $\overline{Q}$  contains a unique point (of  $X_2$ ) collinear with  $x$ . So,  $x$  is collinear with precisely  $3q$  points of  $X_2$ . For every point  $z$  of  $X_3$  collinear with  $x$ , let  $z'$  denote the unique point of  $X_1$  collinear with  $z$ . There are precisely  $q - 1$  points  $z$  of  $X_3$  collinear with  $x$  for which  $z' = y$ , namely the  $q - 1$  points on the line  $xy$  distinct from  $x$  and  $y$ . Now, there are  $3q$  points in  $X_1$  which have distance 1 from  $y$ . If  $u$  is one of these points, then  $x$  and  $u$  have  $q^2 - 1$  common neighbors which are not contained in any quad  $\overline{Q}$  where  $Q$  is a quad of  $\widetilde{X}_1$ . All these points belong to  $X_3$ . If  $u$  is one of the  $q^2$  points of  $X_1$  at distance 2 from  $x$  and not collinear with  $y$ , then  $u$  and  $x$  have  $q^2 - 2$  common neighbors which are not contained in any quad  $\overline{Q}$  where  $Q$  is a quad of  $\widetilde{X}_1$ . All these points belong to  $X_3$ . Now, every point of  $X_3$  collinear with  $x$  is obtained in one of the above-described ways. Hence,  $x$  is collinear with  $(q - 1) + 3q(q^2 - 1) + q^2(q^2 - 2) = q^4 + 3q^3 - 2q^2 - 2q - 1$  points of  $X_3$ . So,  $x$  is collinear with  $(q^5 + q^3 + q) - 1 - 3q - (q^4 + 3q^3 - 2q^2 - 2q - 1) = q^2(q - 1)(q^2 - 2)$  points of  $X_4$ .

Let  $x$  be a point of  $X_4$ . Then  $x$  is collinear with 0 points of  $X_1$ . Every quad  $\overline{Q}$  where  $Q$  is one of the  $3(q + 1)$  quads of  $\widetilde{X}_1$  contains a unique point (of  $X_2$ ) collinear with  $x$ . Hence,  $x$  is collinear with  $3(q + 1)$  points of  $X_2$ . Now,  $x$  lies at distance 2 from precisely  $(q + 1)^2$  points of  $X_1$ . If  $y$  is one these points, then there are three neighbors of  $x$  and  $y$  which belong to  $X_2$  (one in each quad  $\overline{Q}$  where  $Q$  is a quad of  $\widetilde{X}_1$  through  $y$ ). The remaining  $q^2 - 2$  neighbors belong to  $X_3$ . So,  $x$  is collinear with  $(q + 1)^2(q^2 - 2)$  points of  $X_3$ . It follows that  $x$  is collinear with  $q^5 + q^3 + q - 3(q + 1) - (q + 1)^2(q^2 - 2) = q^5 - q^4 - q^3 + q^2 + 2q - 1$  points of  $X_4$ .

So,  $\{X_1, X_2, X_3, X_4\}$  is a regular partition of  $DH(5, q^2)$  with associated coefficient matrix

$$\begin{bmatrix} 3q & 3q(q^2 - 1) & q(q^2 - 1)^2 & 0 \\ q + 1 & q^3 + q - 1 & q(q - 1)(q + 1)^2 & q^2(q - 1)^2(q + 1) \\ 1 & 3q & q^4 + 3q^3 - 2q^2 - 2q - 1 & q^2(q - 1)(q^2 - 2) \\ 0 & 3(q + 1) & (q + 1)^2(q^2 - 2) & q^5 - q^4 - q^3 + q^2 + 2q - 1 \end{bmatrix}.$$

The associated eigenvalues are  $\lambda_0 = q^5 + q^3 + q$  (multiplicity 1),  $\lambda_1 = q^3 + q - 1$  (multiplicity 2) and  $\lambda_2 = -q^2 + q - 1$  (multiplicity 1).

(c) Suppose  $X_1$  is a subspace of  $DH(5, q^2)$  such that  $\widetilde{X}_1 \cong Q(4, q) \times \mathbb{L}_{q+1}$  is isometrically embedded into  $DH(5, q^2)$ . So,  $|X_1| = (q + 1)^2(q^2 + 1)$ . Every quad  $Q$  of  $\widetilde{X}_1$  is contained in a unique quad  $\overline{Q}$  of  $DH(5, q^2)$ . If  $Q$  is one of the  $(q + 1)(q^2 + 1)$  grid-quads of  $\widetilde{X}_1$ ,

then  $\overline{Q}$  contains  $(q+1)(q^3 - q)$  points which are not contained in  $X_1$ . If  $Q$  is one of the  $q+1$   $Q(4, q)$ -quads of  $\widetilde{X}_1$ , then  $\overline{Q}$  contains  $(q+1)(q^3 - q^2)$  points which are not contained in  $X_1$ . Put  $X_2 := \bigcup_Q (\overline{Q} \setminus Q)$  where the union ranges over all  $Q(4, q)$ -quads  $Q$  of  $\widetilde{X}_1$  and  $X_3 := \bigcup_Q (\overline{Q} \setminus Q)$  where the union ranges over all grid-quads  $Q$  of  $\widetilde{X}_1$ . Then  $|X_2| = (q+1) \cdot (q+1)(q^3 - q^2) = q^2(q-1)(q+1)^2$  and  $|X_3| = (q+1)(q^2+1) \cdot (q+1)(q^3 - q) = q(q-1)(q+1)^3(q^2+1)$ . Every point  $x$  of  $X_1$  is contained in  $q+2$  quads  $Q_1, Q_2, \dots, Q_{q+2}$  of  $\widetilde{X}_1$ . One easily verifies that there are  $q(q-1)^2(q+1)$  lines through  $x$  not contained in  $\overline{Q_1} \cup \overline{Q_2} \cup \dots \cup \overline{Q_{q+1}}$ . Now, let  $X_4$  denote the set of points of  $DH(5, q^2)$  not contained in  $X_1 \cup X_2 \cup X_3$  which are collinear with a (necessarily unique) point of  $X_1$ . Then by the above,  $|X_4| = |X_1| \cdot q(q-1)^2(q+1) \cdot q = q^2(q-1)^2(q+1)^3(q^2+1)$ . Since  $|X_1| + |X_2| + |X_3| + |X_4| = (q+1)(q^3+1)(q^5+1)$ , we have counted all the points of  $DH(5, q^2)$ .

Let  $x$  be a point of  $X_1$ . There are  $q+2$  lines  $L$  through  $x$  such that  $L \setminus \{x\}$  is completely contained in  $X_1$ ,  $q(q-1)$  lines  $L$  through  $x$  such that  $L \setminus \{x\}$  is completely contained in  $X_2$ ,  $(q^2-1)(q+1)$  lines through  $x$  such that  $L \setminus \{x\}$  is completely contained in  $X_3$  and  $q(q-1)^2(q+1)$  lines  $L$  through  $x$  such that  $L \setminus \{x\}$  is completely contained in  $X_4$ . So,  $x$  is collinear with  $q(q+2)$  other points of  $X_1$ ,  $q^2(q-1)$  points of  $X_2$ ,  $q(q-1)(q+1)^2$  points of  $X_3$  and  $q^2(q-1)^2(q+1)$  points of  $X_4$ .

Let  $x$  be a point of  $X_2$ . Then there exists a unique  $Q(4, q)$ -quad  $Q$  of  $\widetilde{X}_1$  such that  $x \in \overline{Q} \setminus Q$ . The quad  $Q$  contains  $q^2+1$  points collinear with  $x$  and these are all the points of  $X_1$  collinear with  $x$ . The points of  $X_2$  which are collinear with  $x$  are either contained in  $\overline{Q} \setminus Q$  or in a quad  $\overline{R}$  where  $R$  is one of the  $q$   $Q(4, q)$ -quads of  $\widetilde{X}_1$  which are disjoint from  $Q$ . Each such quad  $\overline{R}$  is disjoint from  $\overline{Q}$  and contains a unique point (of  $X_2$ ) collinear with  $x$ . It follows that  $x$  is collinear with  $(q^2+1)(q-1) + q = q^3 - q^2 + 2q - 1$  points of  $X_2$ . Clearly,  $x$  cannot be collinear with points of  $X_3$ . So,  $x$  is collinear with precisely  $q^5 + q^3 + q - (q^2+1) - (q^3 - q^2 + 2q - 1) = q(q^4 - 1)$  points of  $X_4$ .

Let  $x$  be a point of  $X_3$ . Then there exists a unique grid-quad  $Q$  of  $\widetilde{X}_1$  such that  $x \in \overline{Q} \setminus Q$ . The quad  $Q$  contains  $q+1$  points collinear with  $x$  and these are all the points of  $X_1$  collinear with  $x$ . No point of  $X_2$  is collinear with  $x$ . The points of  $X_3$  which are collinear with  $x$  are either contained in  $\overline{Q} \setminus Q$  or in a quad  $\overline{R}$  where  $R$  is one of the  $q^3$  grid-quads of  $\widetilde{X}_1$  which are disjoint from  $Q$ . Each such quad  $\overline{R}$  is disjoint from  $\overline{Q}$  and contains a unique point (of  $X_3$ ) collinear with  $x$ . It follows that  $x$  is collinear with  $(q^3-1) + q^3 = 2q^3 - 1$  points of  $X_3$  and  $(q^5 + q^3 + q) - (q+1) - (2q^3 - 1) = q^3(q^2 - 1)$  points of  $X_4$ .

Let  $x$  be a point of  $X_4$ . Then  $x$  is collinear with a unique point  $y$  of  $X_1$ . If  $Q$  is one of the  $q$   $Q(4, q)$ -quads of  $\widetilde{X}_1$  not containing  $y$ , then the unique point of  $\overline{Q}$  collinear with  $x$  belongs to  $X_2$ . If  $Q$  is one of the  $(q+1)q^2$  grid-quads of  $\widetilde{X}_1$  not containing  $y$ , then the unique point of  $\overline{Q}$  collinear with  $x$  belongs to  $X_3$ . It follows that  $x$  is collinear with  $q$  points of  $X_2$  and  $(q+1)q^2$  points of  $X_3$ . This implies that  $x$  is collinear with  $(q^5 + q^3 + q) - 1 - q - (q+1)q^2 = q^5 - q^2 - 1$  points of  $X_4$ .



So,  $\{X_1, X_2, X_3, X_4\}$  is a regular partition with associated matrix

$$\begin{bmatrix} q(q+2) & q^2(q-1) & q(q-1)(q+1)^2 & q^2(q-1)^2(q+1) \\ q^2+1 & q^3-q^2+2q-1 & 0 & q(q^4-1) \\ q+1 & 0 & 2q^3-1 & q^3(q^2-1) \\ 1 & q & (q+1)q^2 & q^5-q^2-1 \end{bmatrix}.$$

The associated eigenvalues are  $\lambda_0 = q^5 + q^3 + q$  (multiplicity 1),  $\lambda_1 = q^3 + q - 1$  (multiplicity 2) and  $\lambda_2 = -q^2 + q - 1$  (multiplicity 1).

## 11 Class 9

(a) In De Bruyn and Vandecasteele [12], the so-called valuations of the near octagon  $\mathbb{H}_4$  were classified. This information was subsequently used to obtain some information regarding isometric embeddings of  $\mathbb{H}_4$  into  $DH(7, 4)$ .

Suppose  $X_1$  is a subspace of  $DH(7, 4)$  such that  $\widetilde{X_1} \cong \mathbb{H}_4$  is isometrically embedded into  $DH(7, 4)$ . Then by [12, Section 3], every point of  $DH(7, 4)$  not contained in  $X_1$  is collinear with either 7, 5, 3, 1 or 0 points of  $X_1$ . Let  $X_2, X_3, X_4, X_5$ , respectively  $X_6$ , denote the set of points of  $DH(7, 4)$  not contained in  $X_1$  which are collinear with 7, 5, 3, 1, respectively 0 points of  $X_1$ . It can be proved that  $\{X_1, X_2, X_3, X_4, X_5, X_6\}$  is a regular partition of  $DH(7, 4)$  with associated coefficient matrix

$$\begin{bmatrix} 20 & 10 & 40 & 60 & 40 & 0 \\ 7 & 23 & 0 & 28 & 112 & 0 \\ 5 & 0 & 31 & 40 & 30 & 64 \\ 3 & 2 & 16 & 37 & 48 & 64 \\ 1 & 4 & 6 & 24 & 71 & 64 \\ 0 & 0 & 10 & 25 & 50 & 85 \end{bmatrix}.$$

The computation of the entries of this matrix involves a lot of (sometimes long) technical calculations involving properties of the convex subspaces of  $\mathbb{H}_4$ . We have therefore opted to give only the matrix. The eigenvalues of the regular partition  $\{X_1, X_2, \dots, X_6\}$  are  $\lambda_0 = 170$ ,  $\lambda_1 = 41$  (multiplicity 2) and  $\lambda_2 = 5$  (multiplicity 3).

(b) Let  $X_1$  and  $X_2$  be as in (a). Then by [12, Section 3],  $\widetilde{X_1 \cup X_2} \cong DW(7, 2)$ . It is straightforward to verify that  $\{X_1, X_2\}$  is a regular partition of  $\widetilde{X_1 \cup X_2}$  with associated coefficient matrix

$$\begin{bmatrix} 20 & 10 \\ 7 & 23 \end{bmatrix}.$$

The eigenvalues of the regular partition  $\{X_1, X_2\}$  of  $\widetilde{X_1 \cup X_2} \cong DW(7, 2)$  are  $\lambda'_0 = 30$  and  $\lambda'_1 = 13$ .

(c) In De Bruyn and Vandecasteele [13], the so-called valuations of the near octagon  $\mathbb{G}_4$  were classified. This information was subsequently used to obtain some information regarding isometric embeddings of  $\mathbb{G}_4$  into  $DH(7, 4)$ .

Suppose  $X_1$  is a subspace of  $DH(7, 4)$  such that  $\widetilde{X_1} \cong \mathbb{G}_4$  is isometrically embedded into  $DH(7, 4)$ . Then by [13, Section 6.6], every point of  $DH(7, 4)$  not contained in  $X_1$  is collinear with either 15 or 7 points of  $X_1$ . Let  $X_2$ , respectively  $X_3$ , denote the set of points of  $DH(7, 4)$  not contained in  $X_1$  which are collinear with 15, respectively 7, points of  $X_1$ . Then it can be proved that  $\{X_1, X_2, X_3\}$  is a regular partition of  $DH(7, 4)$  with associated coefficient matrix

$$\begin{bmatrix} 44 & 72 & 54 \\ 15 & 65 & 90 \\ 7 & 56 & 107 \end{bmatrix}.$$

Again, the computation of the entries of this matrix involves some technical calculations involving properties of the convex subspaces of  $\mathbb{G}_4$ . The eigenvalues of the regular partition  $\{X_1, X_2, X_3\}$  are  $\lambda_0 = 170$ ,  $\lambda_1 = 41$  and  $\lambda_2 = 5$ .

## References

- [1] J. Bamberg, M. Law and T. Penttila. Tight sets and  $m$ -ovoids of generalised quadrangles. *Combinatorica* 29 (2009), 1–17.
- [2] A. E. Brouwer, A. M. Cohen and A. Neumaier. *Distance-Regular Graphs*. Springer-Verlag, Berlin, 1989.
- [3] P. J. Cameron. Dual polar spaces. *Geom. Dedicata* 12 (1982), 75–85.
- [4] I. Cardinali, B. De Bruyn and A. Pasini. Locally singular hyperplanes in thick dual polar spaces of rank 4. *J. Combin. Theory Ser. A* 113 (2006), 636–646.
- [5] A. Cossidente, C. Culbert, G. L. Ebert and G. Marino. On  $m$ -ovoids of  $W_3(q)$ . *Finite Fields Appl.* 14 (2008), 76–84.
- [6] A. Cossidente and T. Penttila. Hemisystems on the Hermitian surface. *J. London Math. Soc. (2)* 72 (2005), 731–741.
- [7] B. De Bruyn. *Near polygons*. Birkhäuser, Basel, 2006.
- [8] B. De Bruyn. Isometric full embeddings of  $DW(2n-1, q)$  into  $DH(2n-1, q^2)$ . *Finite Fields Appl.* 14 (2008), 188–200.
- [9] B. De Bruyn. On the intersection of distance- $j$ -ovoids and subpolygons in generalized polygons. *Discrete Math.* 309 (2009), 3023–3031.
- [10] B. De Bruyn and H. Pralle. The hyperplanes of  $DH(5, q^2)$ . *Forum Math.* 20 (2008), 239–264.
- [11] B. De Bruyn and P. Vandecasteele. Valuations and hyperplanes of dual polar spaces. *J. Combin. Theory Ser. A* 112 (2005), 194–211.

- [12] B. De Bruyn and P. Vandecasteele. The valuations of the near octagon  $\mathbb{H}_4$ . *Graphs Combin.* 23 (2007), 601–623.
- [13] B. De Bruyn and P. Vandecasteele. The valuations of the near octagon  $\mathbb{G}_4$ . *Discrete Math.*, to appear.
- [14] A. De Wispelaere and H. Van Maldeghem. Regular partitions of (weak) finite generalized polygons. *Des. Codes Cryptogr.* 47 (2008), 53–73.
- [15] C. D. Godsil. *Algebraic combinatorics*. Chapman and Hall Mathematics Series. Chapman & Hall, New York, 1993.
- [16] C. D. Godsil and G. Royle. *Algebraic graph theory*. Graduate Texts in Mathematics 207. Springer-Verlag, New York, 2001.
- [17] S. E. Payne. Tight pointsets in finite generalized quadrangles. Eighteenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1987). *Congr. Numer.* 60 (1987), 243–260.
- [18] S. E. Payne. Tight pointsets in finite generalized quadrangles. II. Proceedings of the Twenty-first Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1990). *Congr. Numer.* 77 (1990), 31–41.
- [19] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles*. Research Notes in Mathematics 110. Pitman, Boston, 1984.
- [20] H. Pralle. Hyperplanes of dual polar spaces of rank 3 with no subquadrangular quad. *Adv. Geom.* 2 (2002) 107–122.
- [21] H. Pralle and S. Shpectorov. The ovoidal hyperplanes of a dual polar space of rank 4. *Adv. Geom.* 7 (2007), 1–17.
- [22] E. E. Shult. Generalized hexagons as geometric hyperplanes of near hexagons. pp. 229–239 in *Groups, Combinatorics and Geometry* (M. Liebeck, J. Saxl, Eds.), Cambridge University Press, Cambridge, 1992.
- [23] J. Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics 386. Springer, Berlin, 1974.

#### Addresses:

Ilaria Cardinali  
 Department of Engineering  
 University of Siena  
 Via Roma, 56  
 I-53100 Siena, Italy  
 cardinali3@unisi.it

Bart De Bruyn  
Department of Pure Mathematics and Computer Algebra  
Ghent University  
Krijgslaan 281 (S22)  
B-9000 Gent, Belgium  
bdb@cage.ugent.be